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# Constrained optimization of driver control to limit energy consumption

R. Jorge Do Marco<sup>1,2,3</sup>, G. Perrin<sup>1</sup>  
C. Funfschilling<sup>2</sup>, C. Soize<sup>3</sup>

<sup>1</sup> COSYS, Université Gustave Eiffel, Marne-La-Vallée, France

<sup>2</sup>Direction Technologies, Innovation et Projets Groupe, SNCF,  
Saint-Denis, France

<sup>3</sup>MSME UMR 8208, Université Gustave Eiffel, Marne-La-Vallée,  
France

## Abstract

This work aims to optimize the driver control along a track to ensure minimal energy consumption. The focus is on optimizing a control system, which requires a dynamic model to feed an energy model. This will enable the linking of control and consumption while checking operational constraints such as punctuality and safety that apply to the dynamics of the train. In both models it is necessary to determine parameters that are not directly measurable and potentially variable from one trip to another (such as the mass of the passengers). As we have both expert knowledge and real measurements, this work focuses on Bayesian calibration to deduce an *a posteriori* distribution; from this distribution, we will extract the maximum from this *a posteriori* distribution in order to perform deterministic optimization. The conclusion of this work is that energy can be reduced. However, the robustness of the model is not sufficient, since a small variation of variable parameters (passenger mass or wind) could cause the operational constraints to be violated.

**Keywords:** optimization, Bayesian calibration, energy economy, driving assistance, automatic-train operation, model predictive control

# 1 Introduction

Limiting energy consumption is one of the challenges facing rail operators. Of the various ways of meeting these two challenges, it has been decided to focus on the dynamics of the train. This lever has been studied in the past, as done by [1], focusing on the optimization of the speed profile. However, the resulting acceleration profile is not continuous. In addition, the model does not take into account the energy recovery capacity of electric train. As a result, the proposed formulation is not adequate. To overcome these problems, we focus on the driver control. So we want to build a control guide that is valid for a large proportion of trains and that minimizes energy while respecting operational constraints.

Assuming that the set of the parameters describing a given train can be represented by a vector  $\mathbf{z}^*$  and letting  $\mathbb{S}$  be a track, we want to find the control  $u^* \in \mathbb{U}_{adm}$  that minimizes the energy consumed where  $\mathbb{U}_{adm}$  is the set of controls satisfying the operational constraints (punctuality and safety). Note that  $\mathbb{U}_{adm} \subset \mathbb{U}$  where  $\mathbb{U}$  is the set of control functions.

The consumed energy will be written as the integral of the consumed power  $h$  along the track such that

$$u^* \in \arg \min_{u \in \mathbb{U}_{adm}} \left( \int_{\mathbb{S}} h(s; \mathbf{z}, \mathbf{x}, u) ds \right), \quad (1)$$

where the state vector  $\mathbf{x}$  is the solution of a deterministic parametric equation

$$\forall s \in \mathbb{S}, \quad \dot{\mathbf{x}}(s) = f(\mathbf{x}(s), u, \mathbf{z}^*). \quad (2)$$

For a given control  $u$  and a given train  $\mathbf{z}$ , it is assumed that function  $f$  is such that equation (2) has a unique solution written as

$$\mathbf{x}(\cdot; u, \mathbf{z}) : \begin{cases} \mathbb{S} & \rightarrow & \mathbb{U} \\ s & \mapsto & \mathbf{x}(s; u, \mathbf{z}) \end{cases}. \quad (3)$$

In this work, we restrict the context to a single train moving forward on a single track. The interactions between multiple trains are not taken into account. We also assume that there are not any "unexpected event" along the track.

In the next sections, we give a brief overview of the considered models and the underlying hypotheses. We also provide a brief description of the Bayesian calibration process and conclude with a vision of the problem solution.

## 2 Driver control optimization

In this section we will briefly review the methodology used to solve the optimization problem. In the following it is assumed that the train moves from a position 0 to a position  $s_f$  on a given fixed track.

## 2.1 Physical model

The very first point of interest in this work is the definition of a physical model. This physical model will then be evaluated on numerous occasions during its calibration or even during numerical experiments. We therefore need a model that is both sufficiently accurate to represent the phenomena of interest and sufficiently coarse to ensure our ability to calibrate the parameters of this model. In this sense, a full rigid-body dynamic model of the train such as those constructed with Vampire® software [2] will not be considered because of their large number of parameters.

As this work focuses on energy consumption, the longitudinal dynamic model is an appropriate model [3], which has been used by the literature [4, 5]. Under this consideration, the state vector  $\mathbf{x}$  is a spatial function that returns both the position and the speed of the leading car.

According to [4], the driver control can be defined as a function

$$u : [0; s_f] \rightarrow [-1; 1], \quad (4)$$

which models the decision of the driver to use a certain part of the traction / braking capacity of the train such that  $-1$  represents a braking at full capacity (excluding emergency braking) and  $1$  represents traction at full capacity. Note that for electric trains, which have both dynamic braking (motor inversion) and pneumatic braking, algorithms are used to switch from one type of braking to the other, so that it is possible to define two functions  $u_d(s; u)$  and  $u_p(s; u)$  representing the effective control for dynamic braking and the effective control for pneumatic braking. Let  $f_{\text{traction}}(\dot{s})$  be the speed-dependent traction force,  $f_d(\dot{s})$  be the speed-dependent dynamic braking force,  $f_p(\dot{s})$  be the speed-dependent pneumatic braking force and  $f_{\text{driver}}$  the driver force

$$f_{\text{driver}}(s; u, \dot{s}) = \begin{cases} f_{\text{traction}}(\dot{s})u(s) & \text{if } u(s) > 0, \\ f_d(\dot{s})u_d(s; u) + f_p(\dot{s})u_p(s; u) & \text{if } u \leq 0. \end{cases} \quad (5)$$

Apart from the driver force, the train suffers from running resistance, which can be model by using the Davis' law [6] that synthesizes all friction (solid and fluid) phenomena by a function  $f_{\text{davis}}$  taking three coefficients  $a$ ,  $b$ , and  $c$  as parameters such that

$$f_{\text{davis}}(s, \dot{s}; a, b, c) = a\mathbb{1}(\dot{s}) + b\dot{s} + c(\dot{s} - v_{\text{wind}}(s))|\dot{s} - v_{\text{wind}}(s)|, \quad (6)$$

where  $v_{\text{wind}}$  is the longitudinal wind speed of the train that is supposed steady but not uniform and  $\mathbb{1}$  is the characteristic function that returns 1 if  $\dot{s}$  is strictly positive and 0 otherwise.

To refine the model, we consider that each car can have its own contribution especially for curves and slopes in the track. Let the train have  $n$  cars. Let  $\theta(s)$  be the declivity at  $s$ ,  $r(s)$  be the absolute curvature radius at  $s$ ,  $m_i$  be the mass of the  $i$ -th car

from the head car, and  $\delta s_i$  the spacing between the  $i$ -th car and the head car. Then

$$f_{\text{weight}}(s; \mathbf{m}) = g \sum_{i=1}^n m_i \theta (s - \delta s_i) + g \sum_{i=1}^n \frac{k_c m_i}{r (s - \delta s_i)}, \quad (7)$$

where  $g$  is the gravity acceleration and  $k_c$  is a corrective coefficient.

Newton's second law enables us to write

$$k_i \sum_{i=1}^n m_i \ddot{s} = f_{\text{driver}}(s; u, \dot{s}) - f_{\text{davis}}(s, \dot{s}; a, b, c) - f_{\text{weight}}(s; \mathbf{m}), \quad (8)$$

where the mass of the motor cars are known and where the mass of the passenger cars are supposed uniform over the cars such that the distribution of mass is reduced to the knowledge of the total mass denoted by  $m$ .

As we want to work on energy, an energetic model of the train will be depicted by a power balance. Let  $h$  be the consumed power,  $p_t$  be the traction power,  $\eta_t$  be the yield of the traction electro-mechanical chain,  $p_r$  be the recuperated power by motor inversion,  $\eta_r$  the yield of the recuperation power. Then

$$h(s; \cdot, \mathbf{x}, u) = \frac{p_t}{\eta_t} - \eta_r p_r + p_{\text{aux}}, \quad (9)$$

where  $p_{\text{aux}}$  is the auxiliary power assumed to be constant according to [4]. In this work, the yields are taken to be linear functions of the power according to [4] such that

$$p_t = f_{\text{traction}}(\dot{s})u(s), \quad (10)$$

$$p_r = f_d(\dot{s})u_d(s; u), \quad (11)$$

$$\eta_t = a_\eta \frac{p_t}{p_{t,N}} + b_\eta, \quad (12)$$

$$\eta_r = c_\eta \frac{p_r}{p_{r,N}} + d_\eta, \quad (13)$$

where  $a_\eta, b_\eta, c_\eta$ , and  $d_\eta$  are yield coefficients that should be determined but that are constrained by the definition of a yield (*i.e.*  $(\eta_t, \eta_r) \in [0; 1]^2$ ) and  $p_{t,N}$  and  $p_{r,N}$  are nominal power defined to make yield coefficients dimensionless.

We call "train" a vector  $\mathbf{z}$  such that

$$\mathbf{z} = [m \ a \ b \ c \ a_\eta \ b_\eta \ c_\eta \ d_\eta \ p_{\text{aux}}].$$

Only these parameters are considered uncertain and will be calibrated; all other parameters, such as traction capacity, are assumed to be known.

## 2.2 Bayesian calibration

In the following, we assume that we have a collection of measures  $v$  which, for a given unknown train  $\mathbf{z}^*$ , represents a collection of journeys made on possibly different tracks at different times. For each measure, we have access to the elapsed time, the instantaneous speed, and the power consumed. For them, we deduce the instantaneous position and the energy consumed. We also have access to the knowledge of the SNCF experts. From these measures, the problem is to determine the train that best matches the measures while respecting the expert knowledge.

The aim is to combine the two sources of information - an *a priori* knowledge of the train and the measurements - in the most representative way possible, while taking account of the simplified nature of the models. The Bayesian calibration formalism will be used (see [7–10]). The general applied methodology is inspired from those depicted by [11]. The available information is used to obtain the prior probability density function  $p_{\mathbf{z}}^{\text{prior}}$ . A likelihood function  $\mathcal{L}(\mathbf{z}) = p_{Y|\mathbf{z}}(v|\mathbf{z})$  is defined using measures  $v$  which are considered to be realisations of a random variable  $Y$ . Note that the likelihood function will integrate model errors. The Bayes formula gives a relation including the posterior probability density function  $p_{\mathbf{z}}^{\text{post}}$  such that

$$p_{\mathbf{z}}^{\text{post}} \propto \mathcal{L} \times p_{\mathbf{z}}^{\text{prior}}. \quad (14)$$

In this work, we consider that uncertainty arises from the measures (measurement errors) and from the models (model errors). In this way, we define measurement errors on both the speed and the consumed power that are supposed given and we also define the model errors on both force and power. These four errors are assumed to be additive, Gaussian, and centered. The variances of each measurement error are supposed known while the variances of each model error are unknown and denoted by  $\sigma_f^2$  and  $\sigma_p^2$ .

Let  $\mathbf{q}$  be the vector of quantities of interest (here the concatenation of the velocity vector with the consumed energy vector). The model error associated with the simulated vector of quantities of interest  $\mathbf{q}_{\text{simu}}$  appears to be Gaussian, centered in  $\mathbf{q}_{\text{meas}}$  the measured vector of quantities of interest, and with a covariance matrix  $[c_{\text{mod}}](\sigma_f^2, \sigma_p^2)$  depending on the variances of the model errors. The likelihood function can therefore be written as

$$\mathcal{L}(\mathbf{z}) = \frac{\exp\left(-\frac{1}{2}\left\|\left(\mathbf{q}_{\text{simu}} - \mathbf{q}_{\text{meas}}\right)\left([c_{\text{mod}}](\sigma_f^2, \sigma_p^2))^{-\frac{1}{2}}\right\|^2\right)}{(\sqrt{2\pi})^9 \sqrt{\det [c_{\text{mod}}](\sigma_f^2, \sigma_p^2)}}, \quad (15)$$

where  $\|\cdot\|$  is the Euclidean norm.

The model errors being defined by variances  $\sigma_f^2$  and  $\sigma_p^2$ , it is necessary to find those variances. The method used to determine such parameters will not be depicted here.

The prior probability distribution of vector  $\mathbf{z}$  is constructed using the maximal entropy principle (see [12–15]) such that

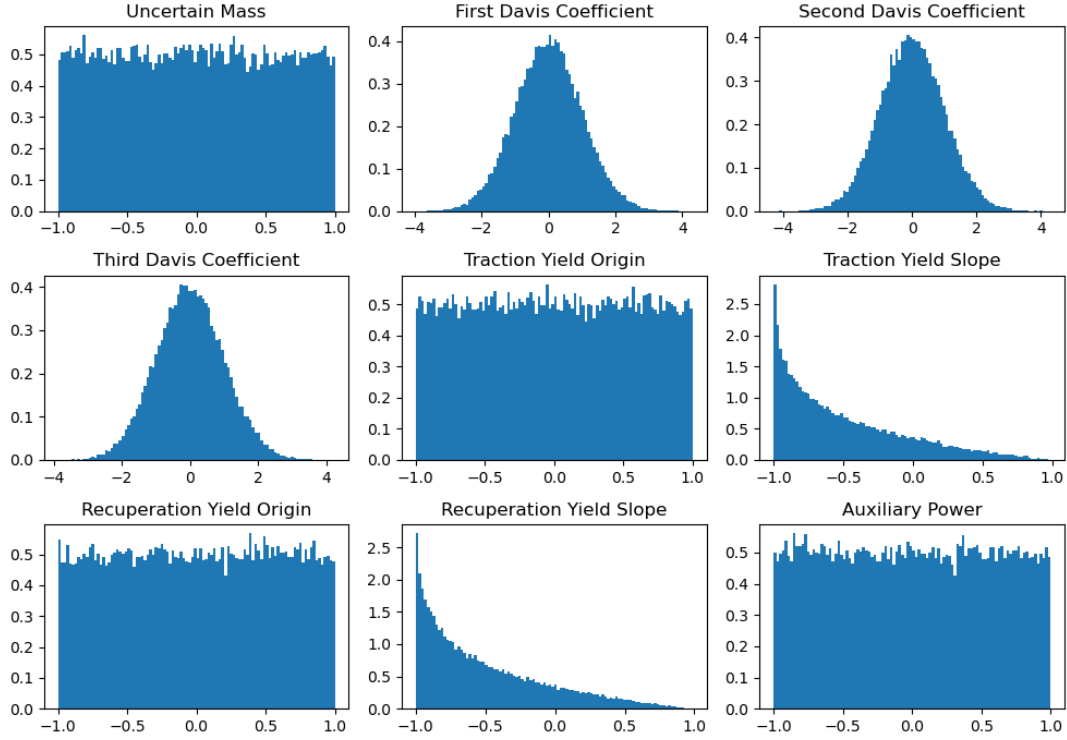


Figure 1: Histogram of the different normalized marginal laws for each parameter of the train  $\mathbf{z}$  for a population of 50 000 realisations

1. The mean and variance of  $A$ ,  $B$ , and  $C$  are known and we know that they should be positive, which yields a Gamma distribution for the prior probability model;
2. The support is known for  $M$ ,  $A_\eta$ ,  $B_\eta$ ,  $C_\eta$ , and  $D_\eta$ , which yields a uniform distribution for the prior probability model.

Note that the yields being constrained, we have constrained the slope of each yield ( $A_\eta/C_\eta$ ) by the origin of each yield ( $B_\eta/D_\eta$ ) such that

$$A_\eta \in [0; 1 - B_\eta], \quad (16)$$

$$C_\eta \in [0; 1 - D_\eta]. \quad (17)$$

For approximating the real train  $\mathbf{z}^*$ , the set of parameters that will be used is the maximum  $\mathbf{z}_{\text{MAP}}$ , of the posterior distribution, defined by,

$$\mathbf{z}_{\text{MAP}} \in \arg \max_{\mathbf{z} \in \mathcal{Z}} (p_{\mathbf{z}}^{\text{post}}(\mathbf{z})). \quad (18)$$

where  $\mathcal{Z}$  is the support of the probability density function  $p_{\mathbf{z}}^{\text{post}}$ . Note that we expect to find a unique  $\mathbf{z}_{\text{MAP}}$ . In the following, we suppose that this problem was solved and that we know  $\mathbf{z}_{\text{MAP}}$ .

## 2.3 Deterministic optimization

Given our access to  $\mathbf{z}_{\text{MAP}}$ , our goal is to determine  $u_{\text{MAP}}$  such that

$$u_{\text{MAP}} \in \arg \min_{u \in \mathcal{U}_{\text{adm}}} \left( \int_0^{s_f} h(s; \mathbf{z}_{\text{MAP}}, \mathbf{x}, u) ds \right). \quad (19)$$

The first step is to define the admissible control set  $\mathcal{U}_{\text{adm}}$ . In our case, a control is said admissible if it allows the train to reach the arrival within the arrival time  $t_f$  and with an admissible arrival speed, while respecting the speed limit during all the travel. Note that the control is supposed spatially defined such that the speed can be null only at the extremities of the track; during simulations, if the tested control return a zero speed in the middle of the track, the simulation will stop with a final position  $s_{-1}$ , which is inferior to  $s_{\text{final}}$ ; this underlines another constraint in the problem. The admissible control set will hereafter be defined as

$$\mathcal{U}_{\text{adm}} = \left\{ u \in \mathcal{U} \left| \begin{array}{l} s_{-1} \geq s_f \\ t(s_f) \leq t_f \\ \forall s \in [0; s_f], \quad \dot{s}(s) \leq v_{\text{lim}}(s) \end{array} \right. \right\}. \quad (20)$$

where  $v_{\text{lim}}$  is the speed limit assumed to be known along the track. To release some constraints in the optimization problem, the track that is considered does not take account for the part of the track that is near the station. Indeed, this part (a few hundred meters before the station) is significantly constrained in terms of speed, to the extent that energy efficiency is no longer the top priority.

In terms of optimization, the problem we face is a derivative-free constrained problem in infinite dimension. The infinite dimension is handled by discretization and interpolation; an infinite-dimensional problem is exchanged for a highly finite-dimensional one. The interpolation process will not be described here. To solve this derivative-free constrained problem in high dimension, we use the CMA-ES algorithm introduced by [16], whose performance is widely recognized as reviewed by [17]. It has also been shown that this algorithm is robust to ill-conditioned problems [18] and to multimodal cost functions [19], two difficulties that may arise from the given formulation.

The constraints will be handled using an augmented Lagrangian [20] formulation adapted to CMA-ES [21]. The tuning of this algorithm is set as proposed by [21].

Using the CMA-ES with augmented Lagrangian allows for solving the initial problem. It is therefore possible to optimize control by using the maximum *a posteriori*  $\mathbf{z}_{\text{MAP}}$  as the parameter set for the train.

This methodology is then applied to the case of a given train on a real track with known variable declivity and curvature. During the journey, the wind speed is also assumed to be known. An arrival time is set. The convergence of the algorithm can be seen in Figure 2. In this figure, each black dot corresponds to a tested command. Each



point cloud corresponds to a normalized constraint or to the normalized cost. The red line on all the constraint graphs corresponds to the boundary of the admissible domain set at zero; a constraint is respected if it lies in the negative. For each one of the curves, three distinct phases can be observed: an initialization phase where the algorithm recalibrates the various parameters, an exploration phase where the algorithm allows itself to explore interesting parts of the inadmissible domain and an exploitation phase where the algorithm effectively converges. Firstly, the observed cost is normalized in relation to a real measurement carried out on a similar route under similar conditions; in the studied case, it is possible to observe a gain of 25% in energy compared with the measurement. In this case, we can also see that the punctuality constraint is restrictive, since the algorithm searches for a solution at the boundary of the admissible domain in terms of punctuality (the optimal command is therefore one that takes all the time it is given to arrive). It should be noted that in this case, the safety constraint of speed limit only comes into play at the final position of the train (maximum and minimum final speed), the constraint being respected throughout the journey.

The output of the algorithm includes the optimized velocity profile together with the optimal control, shown in Figure 3. For each graph, the translucent black lines represent the mean of each generation of the CMA-ES, providing an insight into the intensity of the exploration phase. It is noteworthy that the operational constraints significantly restrict the controls, thus limiting the feasible control space. The final average of the algorithm is represented by the purple curve. An interesting observation is the relative shift between the early iterations (isolated and transparent black curves) and the purple curve. This observation indicates that the algorithm finds it advantageous to achieve higher speeds at the beginning of the trajectory and is willing to accept a lower speed towards the end of the journey, probably due to operational constraints. Similarities between the profile and the literature, such as [1], are observed. Consequently, a distinct initial phase is identified where the control is set to 1, representing maximum traction. This is followed by a free-wheeling phase of the train with intermittent traction phases corresponding to the declivity. This variable declivity is also reflected in the oscillations of the speed profile. The final phase involves braking, which can be divided into two phases: an initial soft braking phase to maximize energy recovery, followed by a more intense braking phase using pneumatic braking.

The presented methodology, as demonstrated in the industrial case study above, enables a substantial reduction in energy consumption during transportation while ensuring compliance with operational constraints. It is possible to identify the different phases that make up the empirically established economic profiles, while taking into account the newly observed energy recovery phenomenon.

### 3 Conclusions

The objective of this work was to solve an optimization problem involving a nonlinear system of coupled equations with uncertain parameters and nonlinear constraints.

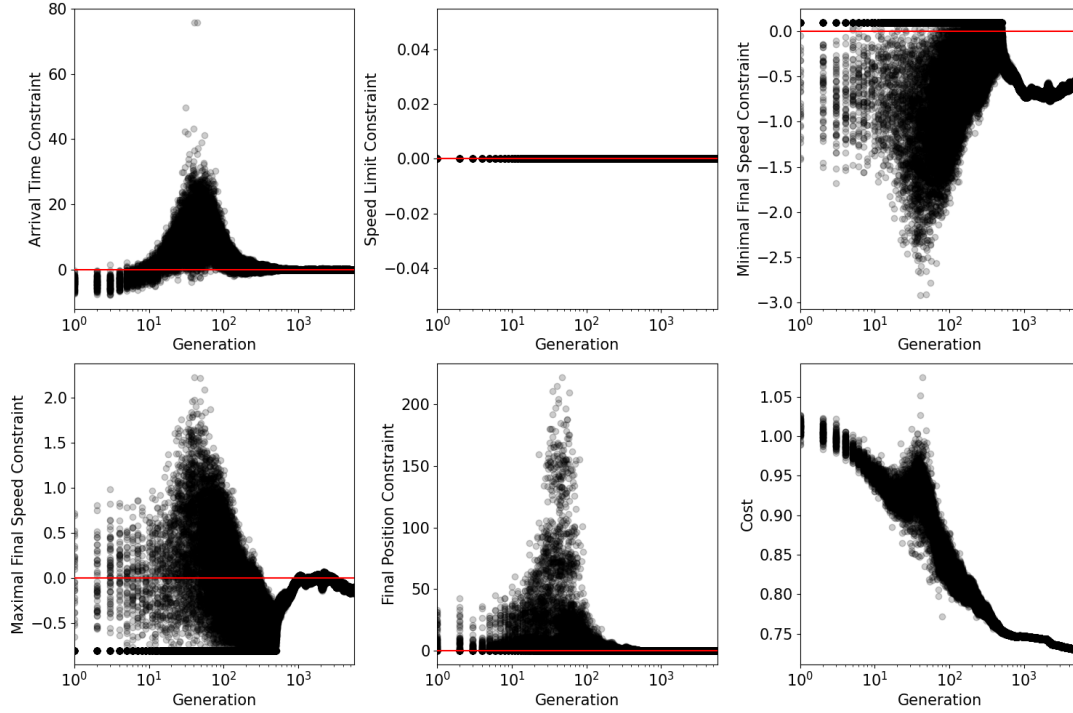


Figure 2: The evolution of cost and constraints as optimization takes place. For each plot, a black dot is a control. The red line is the zero line (limit of acceptability) for each constraint. The cost decreases monotonically, knowing that a cost of one would mean that we have consumed the same energy as the measure.

Overall, the proposed methodology enabled the problem to be solved in three key steps. In the first step, a parametric model was chosen in accordance with the phenomena relevant to the study and the available computational capabilities. In the second step, a calibration step was implemented using a Bayesian formalism in order to make the best use of all the available information. This calibration made it possible to extract the parameters corresponding to the mode of the *a posteriori* distributions, *i.e.* the most probable set of parameters for the train of interest. Finally, a deterministic optimization was carried out on the control system using an adapted evolutionary strategy (CMA-ES) coupled with an augmented Lagrangian algorithm to manage the constraints. The algorithm was then applied to an industrial application.

The results of this method ensure energy optimization of up to several tens of percent. The Bayesian calibration study is also likely to provide us with objective information on the variability of the parameters due to uncertainty.

However, this method does not guarantee the admissibility of the optimal control for the real train (it only guarantees the admissibility and optimality of the control found for the train corresponding to the modal parameter set of the posterior distribution). Furthermore, the calculation time resulting from the use of an evolutionary

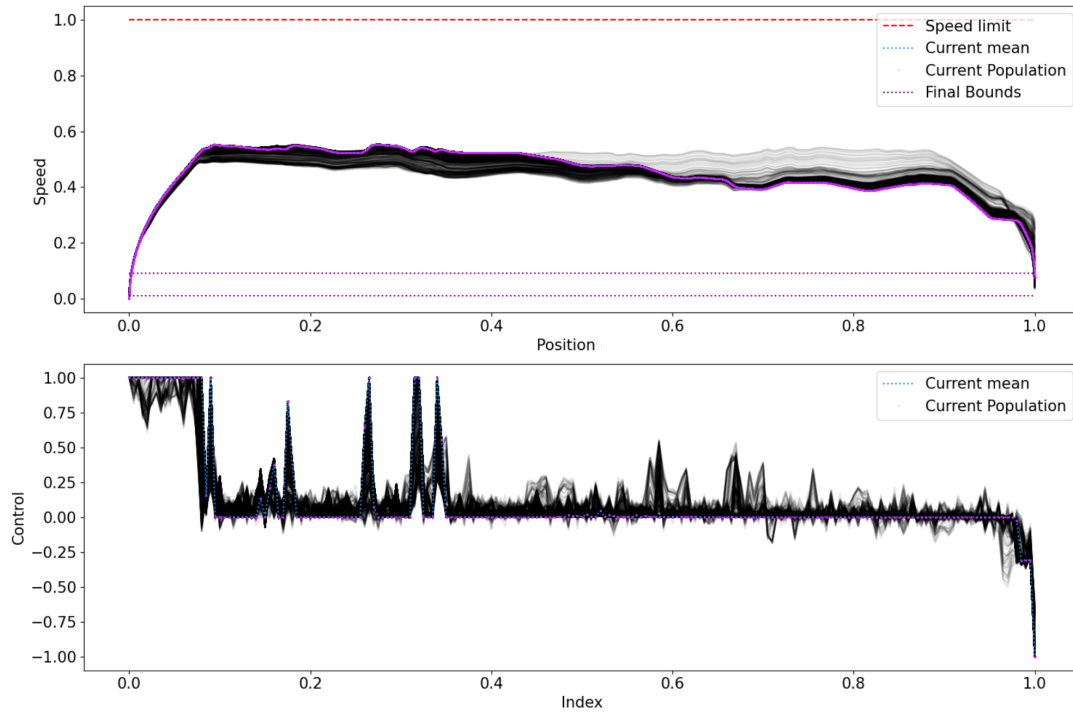


Figure 3: The result of the optimization for one case makes it possible to reconstruct the empirical speed profile of [1] with a full traction phase, a free running phase, and a braking phase. Note that the braking phase is not at full power to ensure maximum energy recovery. Also note that the control is not exactly zero throughout the free-wheeling phase as the environment (wind, slope, curvature) changes.

strategy is problematic insofar as it is not possible to recalculate an optimal control in the event of a last-minute change (missing passenger, wind, etc.).

One strategy could be to apply an algorithm to adapt the command in case of variation, as suggested by [11]. It is also interesting to try to maintain a probabilistic framework, coupled with the use of onboard sensors in the train to provide real-time adaptations.

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