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Abstract

We aim at studying a novel mathematical model associated to a physical phenomenon of infiltration in an homogeneous porous medium. The particularities of our system are connected to the presence of a gravitational acceleration term proportional to the level of saturation, and of a Brownian multiplicative perturbation. Furthermore, the boundary conditions intervene in a Robin manner with the distinction of the behavior along the inflow and outflow respectively. We provide qualitative results of well-posedness, the investigation being conducted through a functional approach.

Keywords: stochastic porous media equations, Robin boundary conditions, maximal monotone operators, Yosida approximation.

MSC2020: 35A01, 60H15, 76S99, 47H05.

1 Introduction

Our contribution primarily focuses on the qualitative study of a fairly involved mathematical model which describes the infiltration of a liquid (water) in an homogeneous porous medium (soil) taking into account the influence of the gravitational acceleration proportional to the level of saturation. In order to provide the readers with a keener insight, we choose to present the physical model prior to further considerations on the different aspects of the state-of-the-art.

1.1 The physical model

Along with delineating the physical model, we choose to present the units of measure for the different notions in paranthesis, such as to guarantee coherence.

First we consider an incompressible fluid with constant density denoted by $\rho\left(\frac{kg}{m^3}\right)$. Then, we consider a reference elementary volume $V_r\left(m^3\right)$ belonging to the flow domain, and we distinguish between V_v the volume of voids and V_w the volume of water in V_r. The flow is said to be unsaturated as long as all pores are not filled with water.
We further introduce the notion of soil moisture as a scalar dimensionless notion by setting

$$\theta = \frac{V_w}{V_r}.$$

As in classical fluid dynamics theory, we combine the Darcy law and the equation of continuity (or mass conservation) in order to get a Richards' type equation which takes into consideration the influence of the gravitational acceleration. The originality of the model consists in the fact that we assumed that the influence of the gravity on the dynamic of the flow is proportional with the moisture. In turn, this translates into further mathematical technical difficulties. The relevance of such models is illustrated shortly after this brief description of the model of interest.

In a classical manner, we invoke Darcy's law written down as

$$q = -\frac{k}{\mu}\left(\nabla p - \rho\theta g i_3\right),$$

where

- q is the fluid volumetric flux (m/s),
- k is the isotropic permeability (m^2),
- μ is the coefficient of viscosity $(kg/(m \cdot s))$,
- p is the pressure $(kg/\left(m \cdot s^2\right))$,
- ρ is the density (kg/m^3),
- θ is the moisture (a dimensionless physical constant)
- gi_3 is the gravity acceleration vector (g being expressed in m/s^2).

On the other hand, the equation of mass conservation yields

$$\frac{\partial(\rho\theta)}{\partial t} + div(\rho q) = f,$$

where ρ, θ and q are as above and f is a water source.
By combining the two equations and keeping in mind that ρ is constant since the fluid is incompressible, it follows that

$$\frac{\partial\theta}{\partial t} - \frac{k}{\mu} div\left(\nabla p - \rho\theta g i_3\right) = f.$$

The attentive reader will have noticed that we have used f instead of a ρ-renormalized source. We have chosen to do so in order to keep notations to a manageable minimum. The hydraulic theory ensures that the water capacity is nothing but the derivative of the moisture with respect to the pressure, i.e.,

$$C(p) = \frac{d\theta}{dp}.$$

As a consequence, one can assume that a primitive of C denoted by C^* is a single-valued positive, twice differentiable, strongly monotonically increasing and concave function such that $\theta = C^*(p)$.

Since it is more convenient to work with the variable θ, we introduce the inverse $(C^*)^{-1}$, in order to obtain an equation of the form

$$\frac{\partial \theta}{\partial t} - \Delta \frac{k}{\mu} (C^*)^{-1} (\theta) + \frac{k\rho g}{\mu} div(\theta i_3) = f.$$

By denoting $\Psi(r) = \frac{k}{\mu} (C^*)^{-1} (r)$ and $K = \frac{k\rho g}{\mu}$, one can rewrite the previous equation in the slightly more general form

$$\frac{\partial \theta}{\partial t} - \Delta \Psi(\theta) + K div(\theta i_3) = f. \tag{1}$$

1.1.1 Numerical illustrations

Let us present some numerical illustrations of the impact of the gravity term in the model. In Figure 1, we consider the porous medium mathematically modeled through $\Psi(r) = r^3$, and we can observe that, for the same source on the top of the domain, the solution is quite different. The humidity is more attracted towards bottom with the gravity term. This type of result motivates the study of the model with the gravity term.

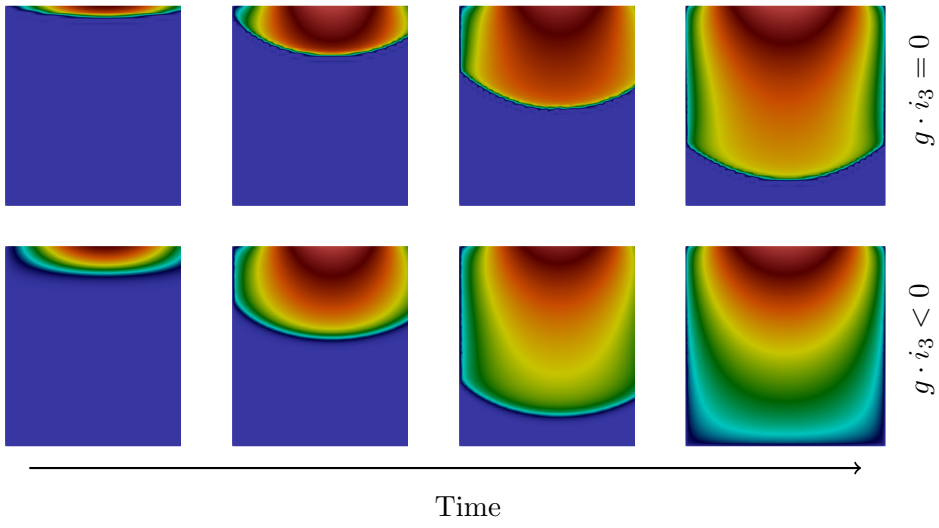

Figure 1: Illustration of the diffusion of the humidity inside the ground during time, from left to right. On the top, without gravity, on the bottom with the gravity term.

1.2 A mathematical model

We are now able to introduce the rigorous mathematical model in the spirit of (1), in which the source f is random and pertaining to a Brownian noise.
To this purpose, we consider a domain $\mathcal{O}$ which is an open bounded subset of $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{O} = \Gamma$. This boundary is splited into two parts denoted respectively by Γ_s (on the

surface) and Γ_u (underground), and such that $\Gamma = \Gamma_s \cup \overline{\Gamma_u}$ and $\Gamma_s \cap \Gamma_u = \emptyset$. This domain extends from the soil surface Γ_s to an underground boundary Γ_u which is supposed to have a variable permeability due to the presence of different types of soils. More precisely, we are interested in the infiltration of the rain from surface boundary of the soil (source which is assumed to be known as the average value of the previous precipitations) and underground water, taking also into account the influence of the gravity on the diffusion process and a linear multiplicative stochastic noise coming from the errors of the measurement.
Our mathematical model is the following

$$(2)\qquad \begin{cases} dX - \Delta\Psi(X)\,dt + K\dfrac{\partial X}{\partial x_3}dt = \sigma(X)\,dW_t, & \text{on } \Omega \times (0,T) \times \mathcal{O}, \\ X(0) = x, & \text{on } \mathcal{O}, \\ (KXi_3 - \nabla\Psi(X))\,\nu = s, & \text{on } \Omega \times (0,T) \times \Gamma_s, \\ (KXi_3 - \nabla\Psi(X))\,\nu - \alpha\Psi(X) = u, & \text{on } \Omega \times (0,T) \times \Gamma_u. \end{cases}$$

The solution $X(\omega, t, \xi)$ of the equation (2) describes the moisture on a path $\omega \in \Omega$. The solution is sought with respect to a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, at each moment $t > 0$ and at each point ξ in space.

The nonlinear operator Ψ contains the physical pieces of information related to the hydraulic conductivity of the soil, while the term $K\dfrac{\partial}{\partial x_3}$ concerns the influence of the gravity on the diffusion process. We denote by ν the outward normal to Γ and by i_3 the unit vector along the axis Ox_3 directed downwards.

The functions s and u are supposed to be known on $\Sigma_s = (0,T) \times \Gamma_s$ and $\Sigma_u = (0,T) \times \Gamma_u$ respectively. Note that the equation on Γ_s expresses the continuity of the normal component of the inflow flux and the equation on Γ_u describes the behavior of the outflow. Finally, the function α gives the variable permeability of the soil on the underground boundary Γ_u.

1.3 State of the art and main contributions

Several different models connected to the porous media diffusion are available. For general results concerning the existence and uniqueness of the solution in the deterministic case for different models including slow, fast and super fast diffusions in saturated or unsaturated flows, the reader is kindly referred to, e.g., [1], [19] and [18].

The stochastic porous media equation has been intensively studied during the recent years. The reader is invited to consuls [2] for a collection of results concerning the existence and uniqueness of the solution for the case with Dirichlet homogeneous boundary conditions. For some critical cases, the papers [8], [9], [14] are a good start. For some further qualitative properties of the solutions, see [4], [12], [13], [15], [6], and for homogenization results see [10] and [11].

To the best of our knowledge, the only type of boundary conditions which was considered previously in the literature for the stochastic porous media equations are of Dirichlet homogeneous nature. On the other hand, the influence of the gravity on the diffusion process was never taken into consideration for the stochastic porous media equations so far, again, to our best knowledge.

In the present work, we develop a new and more realistic physical model that takes into account the influence of the gravity, as well as the different flows on the boundary, through Robin type boundary conditions.

Our model implies several technical difficulties at the mathematical level. Since the usual framework in L^1 or H^{-1} are not appropriate for the Robin-type boundary conditions, one has to

consider a different space whose norm accounts for the different components of the flow, as well as the source on the boundary. For this reason, several results, especially related to the construction and the properties of the stochastic noise, and also to the activity of the porous media operator needed to be adapted to the present case. The estimates and the convergences of the proof are of a rather involved technical nature, because of the presence of several extra terms topping the more complicated spaces and norms previously mentioned.

The paper is organized as follows. In section 2 we explain the fundamental mathematical setting and the standing assumptions employed throughout the paper. The Gelfand triple with the appropriate norms is introduced in Subsection 2.1. The standing assumptions are gathered in Subsection 2.2; they concern the driving porous coefficient, the boundary inputs and the construction of the noise coefficient. The notion of abstract solution is specified in Subsection 2.3, while the remaining Subsection 2.4 concerns the (quasi-) accretivity properties of the abstract nonlinear operator. The main result makes the object of Section 3, through Theorem 5. Its proof is divided into several steps, and consists in two approximations, indexed by $\lambda, \varepsilon > 0$. Showing that the approximating λ, ε-problem is well-posed in L^2 relies on Lemma 6, whose proof is relegated to the Appendix. Passing to the limit as $\varepsilon \to 0$ relies on the estimates in Lemma 7, whose proof is given at the end of Section 3. The remaining arguments are gathered in the proof of the main result.

2 Mathematical tools and assumptions

2.1 A Functional framework

In order to properly define the model, we need to introduce a functional setting which is appropriated to our problem. In particular, since we intend to rely on the Gelfand-triple arguments, we introduce the norms on both the primal and the dual space in order to render the arguments compatible with the homogeneous Robin conditions.

To this purpose, we denote by V the space $H^1(\mathcal{O})$ endowed with the following norm

$$|x|_V = \left(\int_{\mathcal{O}} |\nabla x|^2 \, d\xi + \int_{\Gamma_u} \alpha(\xi) |x|^2 \, d\xi \right)^{1/2},$$

where α is a strictly positive, continuous function, upper bounded and lower bounded away from 0. One can easily check that the norm above is equivalent with the classical Hilbert norm of $H^1(\mathcal{O})$ (see Theorem 2.7 in the Appendix of [18]).
We denote by V' the dual of the space V which is equipped with the scalar product

$$\langle x, \overline{x} \rangle_{V'} = x(\varphi), \quad \forall x, \overline{x} \in V',$$

where $\varphi \in V$ satisfies the boundary value problem

$$\begin{cases} -\Delta \varphi = \overline{x}, & \text{on } \mathcal{O} \\ \dfrac{\partial \varphi}{\partial \nu} + \alpha \varphi = 0, & \text{on } \Gamma_u \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & \text{on } \Gamma_s. \end{cases}$$

Here, $x(\varphi)$ represents the value of the functional x computed at $\varphi \in V$, or the pairing between V and V' which reduces to the scalar product in $L^2(\mathcal{O})$ in the spirit of the Gelfand triple $V \subset L^2(\mathcal{O}) \subset V'$.

We further consider the Laplace operator with Robin boundary conditions, and, more precisely we are interested in the eigenvalues/ eigenfunctions problem

$$\left\{\begin{array}{ll} -\Delta e_j = \lambda_j e_j, & \text{in } \mathcal{O} \\ \dfrac{\partial e_j}{\partial \nu} + \alpha e_j = 0, & \text{on } \Gamma_u \\ \dfrac{\partial e_j}{\partial \nu} = 0, & \text{on } \Gamma_s. \end{array}\right. \tag{3}$$

By the same argument as in the proof of Theorem 6.2 from [3] we obtain the existence of a complete orthonormal system $\{e_k\}_k$ in $L^2(\mathcal{O})$ of eigenfunctions of the Laplace operator with Robin boundary conditions. We denote by $\{\lambda_k\}_k$ the corresponding sequence of real and increasing eigenvalues. We denote by C a general constant that may change from one line to another. The fundamental boundedness properties are gathered in the first part of our Appendix.

2.2 Assumptions

Throughout the paper and unless stated otherwise, we will enforce the following.

1) The function $\Psi : \mathbb{R} \to \mathbb{R}$ is assumed to be a maximal monotone operator in $\mathbb{R}$, continuous, differentiable, increasing, i.e.

$$(\Psi(r) - \Psi(s))(r-s) \geq C_0 |r-s|^2, \quad \forall r, s \in \mathbb{R},$$

for some $C_0 \geq 0$. Furthermore, we ask that

$$\left\{\begin{array}{l} |\Psi(r)| \leq C_1 |r|^m + C_2, \quad \forall r \in \mathbb{R}, \\ j(r) := \int_0^r \Psi(s)\, ds \geq C_3 |r|^{m+1} + C_4 r^2 - C_5, \quad \forall r \in \mathbb{R}, \\ \Psi(0) = 0 \end{array}\right.$$

where $C_i > 0,\ i \in \{1, \ldots, 5\}$ and $1 \leq m$. When $K \neq 0$, we ask that Ψ be strictly increasing, i.e., $C_0 > 0$. We note that since Ψ is increasing, the mean value theorem implies that

$$r\Psi(r) \geq j(r), \quad \forall r \in \mathbb{R}.$$

2) The function $\alpha : \Gamma_u \to [\alpha_m, \alpha_M]$ is positive and continuous such that

$$0 < \alpha_m < \alpha(\xi) < \alpha_M,$$

for all $\xi \in \Gamma_u$.

3) The functions u and s belong to $L^2(\Sigma_u)$ and $L^2(\Sigma_s)$ respectively, and are the traces in each t of some functions $\widetilde{u}$ and $\widetilde{s}$ which belong to $H^1(\mathcal{O})$.

4) The Wiener process W is assumed to be cylindrical on $L^2(\mathcal{O})$ and given by

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t)\, e_j,$$

where $\{\beta_j\}$ is a sequence of mutually independent Brownian motions on the filtered probability $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t\ , \mathbb{P})$ and $\{e_j\}$ is the orthonormal basis in $L^2(\mathcal{O})$ defined by the Robin-Laplace operator.

5) In the noise we assume that Σ is a linear operator from V' to the space of Hilbert-Schmidt operators $L_2\left(L^2(\mathcal{O}); V'\right)$ of the form

$$\Sigma(x)\, h = \sum_{j=1}^{\infty} \mu_j \left(h, e_j\right)_2 x e_j, \quad \forall x \in V',\ h \in L^2(\mathcal{O}).$$

In order to guarantee the Hilbert-Schmidt condition, i.e. $\sum_{j=1}^{\infty} |\Sigma(x)\, e_j|_{V'}^2 < \infty, \quad \forall x \in V'$, we need to enforce the convergence of the serial

$$\sum_{j=1}^{\infty} \mu_j^2 |x e_j|_{V'}^2 \leq C \sum_{j=1}^{\infty} \mu_j^2 \left(1 + \lambda_j^{\frac{d+1}{2}}\right) |x|_{V'}^2.$$

This inequality is guaranteed due to the estimates in the Appendix. For these reasons, we ask that

$$\sum_{j=1}^{\infty} \mu_j^2 \left(1 + \lambda_j^{\frac{d+1}{2}}\right) \leq C.$$

Remark 1 *1. Note that, from the last two assumptions, we get that the form of the noise coefficient is*

$$\Sigma(X)\, dW_t = \sum_{j=1}^{\infty} X e_j \mu_j d\beta_j(t), \quad \forall X \in V',\ t \geq 0.$$

2. The assumption on j in the first point guarantees, in particular, that the range of Ψ covers the whole $\mathbb{R}$.

3. The constant C_5 can be chosen to be positive and this guarantees coverage of cases when Ψ is null on a compact domain, as it is the case, for instance, in Stefan-like problems.

2.3 A notion of solution

In order to explain the choice of the solution for equation (2), we shall first rewrite the equation as a problem set on the space V'.

$$\text{(4)} \qquad \begin{cases} dX(t) + A(X(t))\, dt + F_u(t)\, dt + F_s(t)\, dt = \Sigma(X(t))\, dW_t, \quad \text{for } t \in (0,T), \\ X(0) = x, \end{cases}$$

where the nonlinear operator $A : D(A) \in V' \to V'$ is defined by

$${}_{V'}\langle A(X), \varphi\rangle_V = \int_{\mathcal{O}} \left(\nabla \Psi(X) \cdot \nabla \varphi - KX \frac{\partial \varphi}{\partial x_3}\right) d\xi + \int_{\Gamma_u} \alpha Tr(\Psi(X))\, Tr(\varphi)\, d\sigma,$$

for all $\varphi \in V$. The natural domain is

$$D(A) = \left\{X \in L^2(\mathcal{O});\ \Psi(X) \in H^1(\mathcal{O})\right\}.$$

The reader is invited to note that, since $\Psi(X) \in H^1(\mathcal{O})$, then the trace $Tr(\Psi(X)) \in L^2(\Gamma)$ and therefore the operator A is well defined.

The second and the third operators from the equation (4) belong to $L^2(0,T;V')$, and they are defined by

$$F_u(t)(\varphi) = -\int_{\Gamma_u} u(t)\, Tr(\varphi)\, d\sigma, \quad \forall \varphi \in V,$$

respectively by

$$F_s(t)(\varphi) = -\int_{\Gamma_s} s(t)\, Tr(\varphi)\, d\sigma, \quad \forall \varphi \in V.$$

With the notations above, the equation (4) is well posed as a Cauchy problem in V' and we use this formulation to construct a variational solution by testing the equality against the eigenfunctions, i.e., by requiring

$$\begin{aligned}
&\langle X(t), e_j\rangle_{V'} + \int_0^t \langle A(X(r)), e_j\rangle_{V'} dr + \int_0^t \langle F_u(r), e_j\rangle_{V'} dr + \int_0^t \langle F_s(r), e_j\rangle_{V'} dr \\
&= \langle x, e_j\rangle_{V'} + \int_0^t \langle \Sigma(X(r))\, dW_r, e_j\rangle_{V'}, \quad \forall t \in [0,T],
\end{aligned}$$

$\mathbb{P}$-a.s., and for all e_j from the orthonormal basis specified above.
In order to define a more detailed form of the solution we make some explicit computations which are also going to be useful throughout the paper in order to pass from the nonlinear operators of type A to the underlying real functions Ψ, respectively to K. We have:

$$\langle A(X(r)), e_j\rangle_{V'} = \int_{\mathcal{O}} \nabla\Psi(X)\cdot\nabla\varphi_j d\xi - \int_{\mathcal{O}} KX\frac{\partial\varphi_j}{\partial x_3} d\xi + \int_{\Gamma_u} \alpha Tr(\Psi(X))\, Tr(\varphi_j)\, d\sigma \tag{5}$$

where $\varphi_j = \lambda_j^{-1} e_j$ satisfies the equation

$$-\Delta\varphi_j = e_j, \text{ on } \mathcal{O}, \ \frac{\partial\varphi_j}{\partial\nu} + \alpha\varphi_j = 0, \text{ on } \Gamma_u, \ \frac{\partial\varphi_j}{\partial\nu} = 0, \text{ on } \Gamma_s.$$

The first term on the right-hand side in (5) is treated using Green's formula to get

$$\begin{aligned}
\int_{\mathcal{O}} \nabla\Psi(X)\cdot\nabla\varphi_j d\xi &= -\int_{\mathcal{O}} \Psi(X)\,\Delta\varphi_j d\xi + \int_{\Gamma_u} Tr(\Psi(X))\,\nabla\varphi_j\cdot\nu d\sigma \\
&= \int_{\mathcal{O}} \Psi(X)\, e_j d\xi - \int_{\Gamma_u} \alpha Tr(\Psi(X))\, Tr(\varphi_j)\, d\sigma
\end{aligned}$$

and, therefore,

$$\langle A(X(r)), e_j\rangle_{V'} = \int_{\mathcal{O}} \left(\Psi(X)\, e_j - KX\frac{\partial\varphi_j}{\partial x_3}\right) d\xi. \tag{6}$$

We can now give a precise definition of what the solution of our equation should be seen as.

Definition 2 *Let us consider an initial condition $x \in V'$. A $V'-$valued continuous, $\mathbb{F}-$adapted stochastic process X is called solution to equation (2) on $[0,T]$ if $X \in L^\infty\left(0,T;L^2(\Omega;V')\right)$ and the following equality holds true for all the eigenfunctions e_j:*

$$\begin{aligned}
&\langle X(t), e_j\rangle_{V'} + \int_{\mathcal{O}} \Psi(X)\, e_j d\xi - \int_{\mathcal{O}} KX\frac{\partial\varphi_j}{\partial x_3} d\xi + \int_0^t \langle F_u(r), e_j\rangle_{V'} dr + \int_0^t \langle F_s(r), e_j\rangle_{V'} dr \\
&= \langle x, e_j\rangle_{V'} + \int_0^t \langle \sigma(X(r))\, dW_t, e_j\rangle_{V'}, \quad \forall t \in [0,T],
\end{aligned}$$

$\mathbb{P}$-a.s. and for all e_j from the orthonormal basis defined above and where φ_j satisfies the equation

$$-\Delta\varphi_j = e_j, \text{ on } \mathcal{O}, \ \frac{\partial\varphi_j}{\partial\nu} + \alpha\varphi_j = 0, \text{ on } \Gamma_u, \ \frac{\partial\varphi_j}{\partial\nu} = 0, \text{ on } \Gamma_s. \tag{7}$$

2.4 Quasi-m-accretivity of the operator A

Before proceeding to the main result, we look into monotonicity properties of the operator A previously defined.

Lemma 3 *Under the previous assumptions, the operator A is quasi m-accretive in V' provided that $C_0 > 0$. When $K = C_0 = 0$,the operator remains quasi-accretive.*

Proof. Let μ be a positive real number which is assumed to be large enough. We aim at proving that

$$\langle (\mu I + A) x - (\mu I + A) y, x - y \rangle_{V'} \geq 0, \text{ for all } x, y \in V'.$$

When $C_0 > 0$, we further prove that the range of the operator $\mu I + A$ is full, i.e.,

$$\mathcal{R}(\mu I + A) = V'.$$

We denoted by I the identity operator on V' and by $\mathcal{R}$ the range of the operator $\mu I + A$.

We start with the accretivity of the operator, i.e.

$$\begin{aligned} \langle (\mu I + A) x - (\mu I + A) y, x - y \rangle_{V'} &= \mu |x - y|_{V'}^2 + \int_{\mathcal{O}} (\nabla \Psi(x) - \nabla \Psi(y)) \cdot \nabla \varphi d\xi \\ &\quad - K \int_{\mathcal{O}} (x - y) \frac{\partial \varphi}{\partial x_3} d\xi + \int_{\Gamma_u} \alpha (\Psi(x) - \Psi(y)) \varphi d\sigma, \end{aligned}$$

where

$$-\Delta \varphi = x - y, \text{ on } \mathcal{O}, \ \frac{\partial \varphi}{\partial \nu} + \alpha \varphi = 0, \text{ on } \Gamma_u, \ \frac{\partial \varphi}{\partial \nu} = 0, \text{ on } \Gamma_s. \tag{8}$$

Using Green's formula, we get

$$\begin{aligned} \int_{\mathcal{O}} (\nabla \Psi(x) - \nabla \Psi(y)) \cdot \nabla \varphi d\xi &= -\int_{\mathcal{O}} (\Psi(x) - \Psi(y)) \Delta \varphi d\xi + \int_{\Gamma_u} (\Psi(x) - \Psi(y)) \frac{\partial \varphi}{\partial \nu} d\sigma \\ &= \int_{\mathcal{O}} (\Psi(x) - \Psi(y)) (x - y) d\xi - \int_{\Gamma_u} \alpha (\Psi(x) - \Psi(y)) \varphi d\sigma. \end{aligned}$$

This further yields

$$\begin{aligned} & \langle (\mu I + A) x - (\mu I + A) y, x - y \rangle_{V'} \\ = \ & \mu |x - y|_{V'}^2 + \int_{\mathcal{O}} (\Psi(x) - \Psi(y)) (x - y) d\xi - K \int_{\mathcal{O}} (x - y) \frac{\partial \varphi}{\partial x_3} d\xi \\ \geq \ & \mu |x - y|_{V'}^2 + C_0 |x - y|_{L^2(\mathcal{O})}^2 - K |x - y|_{L^2(\mathcal{O})} \left| \frac{\partial \varphi}{\partial x_3} \right|_{L^2(\mathcal{O})}. \end{aligned}$$

One easily notes that

$$\left| \frac{\partial \varphi}{\partial x_3} \right|_{L^2(\mathcal{O})} = |\langle \nabla \varphi, i_3 \rangle_{\mathbb{R}^3}|_{L^2(\mathcal{O})} \leq |\nabla \varphi|_{L^2(\mathcal{O})}. \tag{9}$$

On the other hand, from (8), it follows that

$$|x - y|_{V'}^2 = |-\Delta \varphi|_{V'}^2 = (-\Delta \varphi)(\gamma) = \langle -\Delta \varphi, \gamma \rangle_2,$$

where γ is the solution of

$$-\Delta \gamma = -\Delta \varphi, \text{ on } \mathcal{O}, \ \frac{\partial \gamma}{\partial \nu} + \alpha \gamma = 0, \text{ on } \Gamma_u, \ \frac{\partial \gamma}{\partial \nu} = 0, \text{ on } \Gamma_s.$$

Since $-\Delta\varphi = x - y$, by the uniqueness of the solution of the previous equation, it follows that $\gamma = \varphi$. Therefore, by invoking once again, Green's formula,

$$|x-y|_{V'}^2 = \langle -\Delta\varphi, \varphi\rangle_2 \geq |\nabla\varphi|_{L^2(\mathcal{O})}^2 .$$

Going back to (9), it follows that

$$\left|\frac{\partial\varphi}{\partial x_3}\right|_{L^2(\mathcal{O})} \leq |x-y|_{V'} , \tag{10}$$

and, as a consequence,

$$\begin{aligned}\langle (\mu I + A)\, x - (\mu I + A)\, y, x - y\rangle_{V'} &\geq \mu\, |x-y|_{V'}^2 + C_0\, |x-y|_{L^2(\mathcal{O})}^2 - K\, |x-y|_{L^2(\mathcal{O})}\, |x-y|_{V'} \\ &\geq 0,\end{aligned} \tag{11}$$

for μ large enough. The reader is invited to note that the same holds true if $K = 0$, for every $\mu > 0$, and regardless of the value of $C_0 \geq 0$. We shall prove now the maximality of the operator under the assumption that $C_0 > 0$. More precisely, we show that for every $g \in V'$ there exist $\theta \in D(A)$ which solve the equation

$$\mu x + Ax = g. \tag{12}$$

Since Ψ is continuous and strongly monotone, increasing on $(-\infty, +\infty)$ and $R(\Psi) = (-\infty, +\infty)$, one establishes that Ψ^{-1} is Lipschitz-continuous, and, therefore, it is continuous from V to $L^2(\mathcal{O})$. (12) can be rewritten as $\mu\Psi^{-1}(y) + \overline{A}(y) = g$, with $\overline{A} : V \to V'$ defined by

$${}_{V'}\left\langle \overline{A}(y), \varphi\right\rangle_V = \int_{\mathcal{O}} \nabla y \cdot \nabla\varphi d\xi - \int_{\mathcal{O}} K\Psi^{-1}(y)\, \frac{\partial\varphi}{\partial x_3} dx + \int_{\Gamma_u} \alpha\Psi^{-1}(y)\, \varphi d\sigma$$

for $\forall\varphi \in V$.

By elementary computations, we can check that $\Psi^{-1} + \overline{A}$ is continuous from V to V', monotone and coercive. By using Minty's theorem (see [3]) we have that $\Psi^{-1} + \overline{A}$ is surjective which implies the existence of an unique solution to equation (12). ■

Remark 4 *We have chosen to give the proof here rather than postponing it to an appendix, since inequalities like* (10) *and the interplay between μ large enough and C_0 and K as emphasized in the proof of (quasi-)accretivity, i.e., through the inequality* (11)*, will play an important part in our subsequent arguments.*

3 The well-posedness result

We are now able to formulate and prove the main result of this paper.

Theorem 5 *For every $x \in L^2(\mathcal{O})$, the equation (2) has a unique solution in the sense of the Definition 2, such that $X \in C_W\left([0,T]; L^2\left(\Omega; L^2(\mathcal{O})\right)\right)$.*

Proof. We have already established that the operator A as defined above is quasi-accretive in V'. The equation (2) can be rewritten, for $\mu > 0$ large enough, in the following equivalent form

$$\begin{cases} dX(t) + (\mu I + A)(X(t))\, dt - \mu X(t)\, dt + F_u(t)\, dt + F_s(t)\, dt = \Sigma(X(t))\, dW_t,\ t \in (0,T), \\ X(0) = x. \end{cases} \tag{13}$$

It is obvious that a solution for $\mu > 0$ is also a solution for some $\mu' = 0$, and vice-versa. Furthermore, the accretivity property of $\mu I + A$ allows one to prove the uniqueness of the solution of (13). Since these arguments are standard and follow from the inequality in (11), we will omit the uniqueness proofs and concentrate on the existence arguments.

We first approximate the operator Ψ by $\widetilde{\Psi}_\lambda = \lambda I + \Psi_\lambda$ where Ψ_λ is the Yosida approximation of Ψ, i.e.

$$\Psi_\lambda(r) = \frac{1}{\lambda}\left(r - (I + \lambda\Psi)^{-1}(r)\right) = \Psi\left((I + \lambda\Psi)^{-1}(r)\right), \quad \forall r \in \mathbb{R}.$$

We denote the resolvent of Ψ by $J_\lambda = (I + \lambda\Psi)^{-1}$. The reader is invited to note that, by a slight abuse of notation, we refer to I as the identity function on $\mathbb{R}$, although, further on, this will also refer to the identity on V'.

Note that $\widetilde{\Psi}_\lambda$ is Lipschitz-continuous and strongly increasing in $\mathbb{R}$. Let us denote by A_λ the operator which is defined as A where one replaces Ψ by its approximation $\widetilde{\Psi}_\lambda$. In particular, note that this operator is quasi m-accretif, owing to Lemma 3. Furthermore, the constant μ can be chosen large enough, and, prior to Step III, we will fix $\lambda > 0$ and pick $\mu = \mu(\lambda)$.

This gives the first approximation of the equation (2)

(14)
$$\begin{cases} dX_\lambda(t) + (\mu I + A_\lambda)(X_\lambda(t))\,dt - \mu X_\lambda(t)\,dt + F_u(t)\,dt + F_s(t)\,dt = \Sigma(X_\lambda(t))\,dW_t,\ t \in (0,T), \\ X(0) = x. \end{cases}$$

In order to get the existence of the solution, we need another approximation. By Lemma 3 the operator $A_\lambda^\mu = \mu I + A_\lambda$ is m-accretive in V' and, therefore, one can take the Yosida approximation of the operator A_λ^μ in V'. For readers convenience, let us recall that

$$J_\lambda^{\mu,\varepsilon} = \left(I + \varepsilon A_\lambda^\mu\right)^{-1}, \quad \forall \varepsilon > 0,$$

is the resolvent of A_λ^μ and

$$A_\lambda^{\mu,\varepsilon} = \frac{1}{\varepsilon}\left(I - J_\lambda^{\mu,\varepsilon}\right) = A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}\right), \quad \forall \varepsilon > 0,$$

is the Yosida approximation of A_λ^μ.

Since the operator $A_\lambda^{\mu,\varepsilon}$ is Lipschitz in V', it follows by standard theory for stochastic equations in Hilbert spaces that the approximating equation

(15)
$$\begin{cases} dX_\lambda^\varepsilon(t) + A_\lambda^{\mu,\varepsilon}(X_\lambda^\varepsilon(t))\,dt - \mu X_\lambda^\varepsilon(t)\,dt + F_u(t)\,dt + F_s(t)\,dt = \sigma(X_\lambda^\varepsilon(t))\,dW_t,\ t \in (0,T), \\ X(0) = x, \end{cases}$$

has a unique strong solution X_λ^ε belonging to $C_W\left([0,T]; L^2(\Omega; V')\right)$ with $A_\lambda^{\mu,\varepsilon}(X_\lambda^\varepsilon(t)) \in C_W\left([0,T]; L^2(\Omega; V')\right)$. We recall that the subscript W specifies adaptedness with respect to the natural filtration, hence with respect to $\mathbb{F}$.

Step I (existence of the solution for the approximating equation in $L^2(\mathcal{O}.)$ In order to have also the existence of the solution in $C_W\left([0,T]; L^2\left(\Omega; L^2(\mathcal{O})\right)\right)$, we shall prove the following preliminary results.

Lemma 6 *The resolvent $y \mapsto J_\lambda^{\mu,\varepsilon}(y) = \left(I + \varepsilon A_\lambda^\mu\right)^{-1}(y)$ is Lipschitz continuous in $L^2(\mathcal{O})$.*

We move the proof to the Appendix for our readers' sake.

Consequently $A_\lambda^{\mu,\varepsilon} = \frac{1}{\varepsilon}\left(I - J_\lambda^{\mu,\varepsilon}\right)$ is Lipschitz continuous in $L^2(\mathcal{O})$ and, therefore, by standard existence theory for stochastic PDEs, for each $x \in L^2(\mathcal{O})$ the equation (15) has a unique solution X_λ^ε in $C_W\left([0,T]; L^2\left(\Omega; L^2(\mathcal{O})\right)\right)$ (see e.g. [16]).

Step II (convergence in ε). In this step, the ε parameter is allowed to vanish, i.e., $\varepsilon \to 0$ and we look into the behavior of the limiting object.

To this purpose, the solution of the approximating equation is written as

$$\begin{aligned} &\langle X_\lambda^\varepsilon(t), e_j\rangle_{V'} + \int_0^t \left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right), e_j\right\rangle_{V'} dr \\ &- \mu\int_0^t \langle X_\lambda^\varepsilon(r), e_j\rangle_{V'} dr + \int_0^t \langle F_u(r), e_j\rangle_{V'} dr + \int_0^t \langle F_s(r), e_j\rangle_{V'} dr \\ &= \langle x, e_j\rangle_{V'} + \sum_{k=1}^\infty \mu_k \int_0^t \langle X_\lambda^\varepsilon(r) e_k, e_j\rangle_{V'} d\beta_k(r), \quad \forall t \in [0,T], \end{aligned} \tag{16}$$

$\mathbb{P}$-a.s., and for all e_j from the orthonormal basis formed by the eigen-functions of the Robin-Laplace operator defined in (3).

Simple computations yield, as before,

$$\begin{aligned} \left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right), e_j\right\rangle_{V'} &= \left\langle \mu J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right), e_j\right\rangle_{V'} + \left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), e_j\right\rangle_{V'} \\ &= \left\langle \mu J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right), e_j\right\rangle_{V'} + \int_{\mathcal{O}} \widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right) e_j d\xi - \int_{\mathcal{O}} K J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) \frac{\partial \varphi_j}{\partial x_3} d\xi. \end{aligned}$$

where φ_j satisfies (7). The solution (16) of the approximating equation is rewritten as follows

$$\begin{aligned} &\langle X_\lambda^\varepsilon(t), e_j\rangle_{V'} + \mu\int_0^t \left\langle J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right), e_j\right\rangle_{V'} dr \\ &+ \lambda\int_0^t \int_{\mathcal{O}} J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right) e_j d\xi dr + \int_0^t \int_{\mathcal{O}} \Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right) e_j d\xi dr \\ &- K\int_0^t \int_{\mathcal{O}} J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right) \frac{\partial \varphi_j}{\partial x_3} d\xi dr - \mu\int_0^t \langle X_\lambda^\varepsilon(r), e_j\rangle_{V'} dr \\ &+ \int_0^t \langle F_u(r), e_j\rangle_{V'} dr + \int_0^t \langle F_s(r), e_j\rangle_{V'} dr \\ &= \langle x, e_j\rangle_{V'} + \sum_{k=1}^\infty \mu_k \int_0^t \langle X_\lambda^\varepsilon(r) e_k, e_j\rangle_{V'} d\beta_k(r), \quad \forall t \in [0,T]. \end{aligned} \tag{17}$$

Passing to the limit relies on the following preliminary results.

Lemma 7 *For $x \in L^2(\mathcal{O})$the following convergence results hold true as $\varepsilon \to 0$.*

$$\begin{aligned} X_\lambda^\varepsilon &\rightharpoonup X_\lambda \text{ weakly in } L^\infty\left(0,T; L^2\left(\Omega; L^2(\mathcal{O})\right)\right), \\ X_\lambda^\varepsilon &\to X_\lambda \text{ strongly in } L^\infty\left(0,T; L^2\left(\Omega; V'\right)\right), \\ \widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right) &\to \eta \text{ strongly in } L^2\left(0,T; L^2\left(\Omega; L^2(\mathcal{O})\right)\right), \end{aligned}$$

where X_λ is a solution to the equation (14). Furthermore, for a constant $C(\lambda)$ independent of $\varepsilon > 0$,

$$\mathbb{E}\int_0^T \left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 dr \le C(\lambda). \tag{18}$$

To facilitate the reading, the proof of the Lemma is postponed after the Theorem.

In order to pass to the limit in (17) we have first to note that

$$\left|X_\lambda^\varepsilon - J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2 \leq \varepsilon \left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2$$

which implies, by (18), that

$$J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) \to X_\lambda \text{ strongly in } L^2\left(0,T;L^2\left(\Omega;V'\right)\right).$$

On the other hand, one proves that

$$J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) \longrightarrow X_\lambda \text{ strongly in } L^2\left(0,T;L^2\left(\Omega;L^2\left(\mathcal{O}\right)\right)\right),$$

see (31) in the proof of Lemma 7.

Finally, since the map Ψ_λ is maximal monotone in $L^2\left(0,T;L^2\left(\Omega;L^2\left(\mathcal{O}\right)\right)\right)$, it is weakly-strongly closed and, therefore,

$$\eta = \Psi_\lambda\left(X_\lambda\right).$$

One can now pass to the limit in (17) to get

$$\begin{aligned}
&\left\langle X_\lambda\left(t\right),e_j\right\rangle_{V'} + \lambda\int_0^t\int_{\mathcal{O}} X_\lambda\left(r\right)e_j d\xi dr + \int_0^t\int_{\mathcal{O}}\Psi_\lambda\left(X_\lambda\left(r\right)\right)e_j d\xi dr \\
&- K\int_0^t\int_{\mathcal{O}} X_\lambda\left(r\right)\frac{\partial\varphi_j}{\partial x_3}d\xi dr + \int_0^t\left\langle F_u\left(r\right),e_j\right\rangle_{V'}dr + \int_0^t\left\langle F_s\left(r\right),e_j\right\rangle_{V'}dr \\
&= \left\langle x,e_j\right\rangle_{V'} + \sum_{k=1}^\infty \mu_k\int_0^t\left\langle X_\lambda\left(r\right)e_k,e_j\right\rangle_{V'}d\beta_k\left(r\right), \quad \forall t\in[0,T],
\end{aligned} \tag{19}$$

where φ_j satisfies (7).

At this point, we emphasize, once again, that μ plays a purely fictitious part in the sense that the equation is the same with some $\mu'=0$ and, from now on, we concentrate on this latter formulation.

Step III a. (weak convergences as $\lambda\to 0$)

In order to pass to the limit for $\lambda\to 0$, one applies Itô's formula with the function $|\cdot|_{V'}$, on $[0,t]$ to the V'-valued Itô process X_λ, and get

$$\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_\lambda\left(t\right)\right|_{V'}^2 + \lambda\mathbb{E}\int_0^t\left|X_\lambda\left(r\right)\right|_{V'}^2 dr + \mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi_\lambda\left(X_\lambda\left(r\right)\right)X_\lambda\left(r\right)d\xi dr \\
&-K\mathbb{E}\int_0^t\int_{\mathcal{O}} X_\lambda\left(r\right)\frac{\partial\varphi_\lambda}{\partial x_3}d\xi dr + \mathbb{E}\int_0^t\left\langle F_u\left(r\right)+F_s\left(r\right),X_\lambda\left(r\right)\right\rangle_{V'}dr \\
= \;& \frac{1}{2}\left|x\right|_{V'}^2 + \mathbb{E}\sum_{k=1}^\infty\mu_k\int_0^t\left|X_\lambda\left(r\right)e_k\right|_{V'}^2 dr,
\end{aligned}$$

where φ_λ is the solution to

$$-\Delta\varphi_\lambda = X_\lambda, \text{ on } \mathcal{O}, \;\; \frac{\partial\varphi_\lambda}{\partial\nu}+\alpha\varphi_\lambda = 0, \text{ on } \Gamma_u, \;\; \frac{\partial\varphi_\lambda}{\partial\nu}=0, \text{ on } \Gamma_s.$$

Keeping in mind that, for some constant C independent of λ,

$$\begin{aligned}
\left|\frac{\partial\varphi_\lambda}{\partial x_3}\right|_2 \;&\leq\; C\left|X_\lambda\right|_{V'} \text{ and } \left|X_\lambda e_k\right|_{V'}\leq C \\
pr1+\lambda_k^{\frac{d+1}{2}}\left|X_\lambda\right|_{V'},
\end{aligned}$$

(see Appendix), and due to the assumptions on F_u and F_s, one has

$$
\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_\lambda(t)\right|_{V'}^2+\lambda\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_{V'}^2dr+\mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi_\lambda\left(X_\lambda(r)\right)X_\lambda(r)\,d\xi dr\\
\leq& C\left(1+|x|_{V'}^2\right)+C\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_{V'}^2dr+\frac{C_4}{4}\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_2^2dr.
\end{aligned}
\tag{20}
$$

On the other hand owing to the assumptions on $j(\cdot)$,

$$
\begin{aligned}
&\mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi_\lambda\left(X_\lambda\right)X_\lambda d\xi dr=\mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi\left(J_\lambda\left(X_\lambda\right)\right)\left(X_\lambda+J_\lambda\left(X_\lambda\right)-J_\lambda\left(X_\lambda\right)\right)d\xi dr\\
=\;&\mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi\left(J_\lambda\left(X_\lambda\right)\right)J_\lambda\left(X_\lambda\right)d\xi dr+\mathbb{E}\int_0^t\int_{\mathcal{O}}\Psi_\lambda\left(X_\lambda\right)\left(X_\lambda-J_\lambda\left(X_\lambda\right)\right)d\xi dr\\
\geq\;&\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(C_3\left|J_\lambda\left(X_\lambda\right)\right|^{m+1}+C_4\left|J_\lambda\left(X_\lambda\right)\right|^2-C_5\right)d\xi dr+\frac{1}{\lambda}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|\left(X_\lambda-J_\lambda\left(X_\lambda\right)\right)\right|^2d\xi dr.
\end{aligned}
$$

Going back to (20), one gets

$$
\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_\lambda(t)\right|_{V'}^2+\lambda\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_{V'}^2dr+\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(C_3\left|J_\lambda\left(X_\lambda\right)\right|^{m+1}+C_4\left|J_\lambda\left(X_\lambda\right)\right|^2-C_5\right)d\xi dr\\
&+\frac{1}{\lambda}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\lambda-J_\lambda\left(X_\lambda\right)\right|^2d\xi dr\\
\leq\;&C\left(1+|x|_{V'}^2\right)+C\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_{V'}^2dr\\
&+\frac{C_4}{2}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\lambda(r)-J_\lambda\left(X_\lambda\right)\right|^2d\xi dr+\frac{C_4}{2}\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|J_\lambda\left(X_\lambda\right)\right|^2d\xi dr.
\end{aligned}
$$

For λ sufficiently small and owing to Gronwall's inequality,

$$
\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_\lambda(t)\right|_{V'}^2+\lambda\mathbb{E}\int_0^t\left|X_\lambda(r)\right|_{V'}^2dr+\mathbb{E}\int_0^t\int_{\mathcal{O}}\left(C_3\left|J_\lambda\left(X_\lambda\right)\right|^{m+1}+\frac{C_4}{2}\left|J_\lambda\left(X_\lambda\right)\right|^2\right)d\xi dr\\
&+\left(\frac{1}{\lambda}-\frac{C_4}{2}\right)\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\lambda-J_\lambda\left(X_\lambda\right)\right|^2d\xi dr\leq C\left(1+|x|_{V'}^2\right),
\end{aligned}
$$

and, therefore,

$$
\begin{aligned}
&\left\{X_\lambda\right\}\text{ is bounded in }L^\infty\left(0,T;L^2\left(\Omega;V'\right)\right)\\
&\left\{J_\lambda\left(X_\lambda\right)\right\}\text{ is bounded in }L^{m+1}\left((0,T)\times\Omega\times\mathcal{O}\right).
\end{aligned}
$$

From the assumptions on the upper bounds on Ψ,

$$
\left\{\Psi\left(J_\lambda\left(X_\lambda\right)\right)\right\}\text{ is bounded in }L^{\frac{m+1}{m}}\left((0,T)\times\Omega\times\mathcal{O}\right).
$$

From the upper bounds on $\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|X_\lambda-J_\lambda\left(X_\lambda\right)\right|^2d\xi dr$, one deduces that

$$
\left\{X_\lambda\right\}\text{ is bounded in }L^2\left((0,T)\times\Omega\times\mathcal{O}\right).
$$

To summarize, one establishes the following weak convergences

$$
\begin{aligned}
X_\lambda\;&\rightharpoonup\;X\text{ weakly in }L^\infty\left(0,T;L^2\left(\Omega;V'\right)\right),\\
&\qquad\text{and }L^2\left((0,T)\times\Omega\times\mathcal{O}\right),\\
J_\lambda\left(X_\lambda\right)\;&\rightharpoonup\;X\text{ weakly in }L^{m+1}\left((0,T)\times\Omega\times\mathcal{O}\right),\\
\Psi\left(J_\lambda\left(X_\lambda\right)\right)\;&\rightharpoonup\;\eta\text{ weakly in }L^{\frac{m+1}{m}}\left((0,T)\times\Omega\times\mathcal{O}\right).
\end{aligned}
$$

Step III b. (strong convergence in $L^{\infty}\left(0,T;L^{2}\left(\Omega;V'\right)\right)$ and conclusion.)
In order to conclude the proof one still has to show that the limit of $\Psi\left(J_{\lambda}\left(X_{\lambda}\right)\right)$ can be identified with $\Psi\left(X\right)$.
Since the operator Ψ is maximal monotone in the duality pair

$$\left(L^{m+1}\left(\left(0,T\right)\times\Omega\times\mathcal{O}\right),L^{\frac{m+1}{m}}\left(\left(0,T\right)\times\Omega\times\mathcal{O}\right)\right),$$

it is sufficient to show that

$$\liminf_{\lambda\to0}\mathbb{E}\int_{0}^{t}\int_{\mathcal{O}}\Psi\left(J_{\lambda}\left(X_{\lambda}\right)\right)J_{\lambda}\left(X_{\lambda}\right)d\xi dr\leq\mathbb{E}\int_{0}^{t}\int_{\mathcal{O}}\eta Xd\xi dr.$$

To this end, it suffices to show the strong convergence of $\{X_{\lambda}\}$ in $L^{\infty}\left(0,T;L^{2}\left(\Omega;V'\right)\right)$. One employs Itô's formula for the squared norm in V' on $[0,t]$, and gets

$$\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_{\lambda}\left(t\right)-X_{\lambda'}\left(t\right)\right|_{V'}^{2}+\mathbb{E}\int_{0}^{t}\left\langle\lambda X_{\lambda}\left(r\right)-\lambda'X_{\lambda'}\left(r\right),X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right\rangle_{V'}dr\\
&+\mathbb{E}\int_{0}^{t}\int_{\mathcal{O}}\left(\Psi_{\lambda}\left(X_{\lambda}\left(r\right)\right)-\Psi_{\lambda'}\left(X_{\lambda'}\left(r\right)\right)\right)\left(X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right)d\xi dr\\
&-K\mathbb{E}\int_{0}^{t}\int_{\mathcal{O}}\left(X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right)\frac{\partial\varphi_{\lambda,\lambda'}}{\partial x_{3}}d\xi dr\\
&+\mathbb{E}\int_{0}^{t}\left\langle F_{u}\left(r\right)+F_{s}\left(r\right),X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right\rangle_{V'}dr=\mathbb{E}\sum_{k=1}^{\infty}\mu_{k}\int_{0}^{t}\left|\left(X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right)e_{k}\right|_{V'}^{2}dr,
\end{aligned}$$

where $\varphi_{\lambda,\lambda'}$ is the solution to

$$-\Delta\varphi_{\lambda,\lambda'}=X_{\lambda}-X_{\lambda'},\text{ on }\mathcal{O},\ \frac{\partial\varphi_{\lambda,\lambda'}}{\partial\nu}+\alpha\varphi_{\lambda,\lambda'}=0,\text{ on }\Gamma_{u},\ \frac{\partial\varphi_{\lambda,\lambda'}}{\partial\nu}=0,\text{ on }\Gamma_{s}.$$

By recalling that, for some constant C independent of λ,

$$\left|\frac{\partial\varphi_{\lambda,\lambda'}}{\partial x_{3}}\right|_{2}\leq C\left|X_{\lambda}-X_{\lambda'}\right|_{V'}\text{ and }\left|\left(X_{\lambda}-X_{\lambda'}\right)e_{k}\right|_{V'}\leq C\left(1+\lambda_{k}^{\frac{d+1}{2}}\right)\left|X_{\lambda}-X_{\lambda'}\right|_{V'},$$

and from the assumptions on F_{u} and F_{s}, it follows that

$$\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_{\lambda}\left(t\right)-X_{\lambda'}\left(t\right)\right|_{V'}^{2}+\mathbb{E}\int_{0}^{t}\left\langle\lambda X_{\lambda}\left(r\right)-\lambda'X_{\lambda'}\left(r\right),X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right\rangle_{V'}dr\\
&+\mathbb{E}\int_{0}^{t}\int_{\mathcal{O}}\left(\Psi_{\lambda}\left(X_{\lambda}\left(r\right)\right)-\Psi_{\lambda'}\left(X_{\lambda'}\left(r\right)\right)\right)\left(X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right)d\xi dr\\
\leq&C\mathbb{E}\int_{0}^{t}\left|X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right|_{V'}^{2}dr+\frac{C_{0}}{8}\mathbb{E}\int_{0}^{t}\left|X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right|_{2}^{2}dr.
\end{aligned}\tag{21}$$

As before

$$\begin{aligned}
&\left(\Psi_{\lambda}\left(X_{\lambda}\right)-\Psi_{\lambda'}\left(X_{\lambda'}\right)\right)\left(X_{\lambda}-X_{\lambda'}\right)\\
=&\left(\Psi_{\lambda}\left(X_{\lambda}\right)-\Psi_{\lambda'}\left(X_{\lambda'}\right)\right)\left(J_{\lambda}\left(X_{\lambda}\right)-J_{\lambda'}\left(X_{\lambda'}\right)\right)\\
&+\left(\Psi_{\lambda}\left(X_{\lambda}\right)-\Psi_{\lambda'}\left(X_{\lambda'}\right)\right)\left(\lambda\Psi_{\lambda}\left(X_{\lambda}\right)-\lambda'\Psi_{\lambda'}\left(X_{\lambda'}\right)\right)\\
\geq&C_{0}\left|J_{\lambda}\left(X_{\lambda}\right)-J_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}-2\left(\lambda+\lambda'\right)\left(\left|\Psi_{\lambda}\left(X_{\lambda}\right)\right|^{2}+\left|\Psi_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\left|X_{\lambda}\left(r\right)-X_{\lambda'}\left(r\right)\right|^{2}&\leq3\left|X_{\lambda}-J_{\lambda}\left(X_{\lambda}\right)\right|^{2}+3\left|X_{\lambda'}-J_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}+3\left|J_{\lambda}\left(X_{\lambda}\right)-J_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}\\
&=3\lambda^{2}\left|\Psi_{\lambda}\left(X_{\lambda}\right)\right|^{2}+3\left(\lambda'\right)^{2}\left|\Psi_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}+3\left|J_{\lambda}\left(X_{\lambda}\right)-J_{\lambda'}\left(X_{\lambda'}\right)\right|^{2}.
\end{aligned}$$

Going back to (21), one concludes, for $0 < \lambda, \lambda' < 1$,

$$\begin{aligned}
&\mathbb{E}\frac{1}{2}\left|X_\lambda(t) - X_{\lambda'}(t)\right|_{V'}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} \frac{C_0}{2}\left|J_\lambda(X_\lambda) - J_{\lambda'}(X_{\lambda'})\right|^2 d\xi dr \\
\leq\ & C\mathbb{E}\int_0^t \left|X_\lambda(r) - X_{\lambda'}(r)\right|_{V'}^2 dr \\
&+C\left(\lambda' + \lambda\right)\mathbb{E}\int_0^t \left[\int_{\mathcal{O}} \left(\left|\Psi_{\lambda'}(X_{\lambda'})\right|^2 + \left|\Psi_\lambda(X_\lambda)\right|^2\right) d\xi + \left|X_\lambda(r)\right|_{V'}^2 + \left|X_{\lambda'}(r)\right|_{V'}^2\right] dr.
\end{aligned}$$

and the strong convergence follows from Gronwall's inequality and the aforementioned estimates for the right-hand terms.

Finally, by the same argument as in [5], we get that $\eta = \Psi(X)$ and the proof is complete. ■

In the previous result, during the second Step, we have used the convergence stated in Lemma 7. Let us now prove those statements.

Proof of Lemma 7. Step 1 (estimates in V'). First, we apply Itô's formula to the function $|\cdot|_{V'}^2$ on $[0,T]$ with the V'-valued diffusion X_λ^ε. We get

$$\begin{aligned}
&\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_{V'}^2 + 2\mathbb{E}\int_0^t \left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right), X_\lambda^\varepsilon(r)\right\rangle_{V'} dr - 2\mu\mathbb{E}\int_0^t \left|X_\lambda^\varepsilon(r)\right|_{V'}^2 dr \\
&+2\mathbb{E}\int_0^t \left\langle F_u(r), X_\lambda^\varepsilon(r)\right\rangle_{V'} dr + 2\mathbb{E}\int_0^t \left\langle F_s(r), X_\lambda^\varepsilon(r)\right\rangle_{V'} dr \leq |x|_{V'}^2 + C\mathbb{E}\int_0^t \left|X_\lambda^\varepsilon(r)\right|_{V'}^2 dr.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
\left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right), X_\lambda^\varepsilon\right\rangle_{V'} &= \left\langle A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_{V'} + \frac{1}{\varepsilon}\left|X_\lambda^\varepsilon - J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2 \\
&= \mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2 + \left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_{V'} + \frac{1}{\varepsilon}\left|X_\lambda^\varepsilon - J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2 \geq 0.
\end{aligned}$$

By using Gronwall's inequality, one has

(22)

$$\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_{V'}^2 + 2\mathbb{E}\int_0^t \mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 dr + 2\mathbb{E}\int_0^t \frac{1}{\varepsilon}\left|X_\lambda^\varepsilon(r) - J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 dr \leq C\left(1 + |x|_{V'}^2\right),$$

for $\forall t \in [0,T]$, where C is a constant independent of ε and λ.

Step 2 (estimates in $L^2(\mathcal{O})$). We continue with applying Itô's formula to the function $|\cdot|_2 := |\cdot|_{L^2(\mathcal{O})}$, to the $L^2(\mathcal{O})$-diffusion X_λ^ε to get

$$\begin{aligned}
&\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_2^2 + 2\mathbb{E}\int_0^t {}_{V'}\left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right), X_\lambda^\varepsilon(r)\right\rangle_V dr - 2\mu\mathbb{E}\int_0^t \left|X_\lambda^\varepsilon(r)\right|_2^2 dr \\
&+2\mathbb{E}\int_0^t \left\langle F_u(r) + F_s(r), X_\lambda^\varepsilon(r)\right\rangle_2 dr \leq |x|_2^2 + C\mathbb{E}\int_0^t \left|X_\lambda^\varepsilon(r)\right|_2^2 dr.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
{}_{V'}\left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right), X_\lambda^\varepsilon\right\rangle_V &= {}_{V'}\left\langle A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) + \varepsilon A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right\rangle_V \\
&= \mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_2^2 + {}_{V'}\left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_V + \varepsilon\left|A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_2^2,
\end{aligned}$$

Gronwall's inequality yields that

(23)

$$\mathbb{E}\left|X_\lambda^\varepsilon(t)\right|_2^2 + 2\mathbb{E}\int_0^t \left(\mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_2^2 + {}_{V'}\left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right), J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right\rangle_V\right) dr \leq C\left(1 + |x|_2^2\right),$$

for $\forall t \in [0,T]$, where C is a constant independent of ε and λ.

Step 3 (estimates of $\mathbb{E}\int_0^t \left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 dr$).

One begins with writing

$$\begin{aligned}\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 &= \left\langle \mu J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) + A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), \mu J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) + A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right\rangle_{V'} \\ &\leq 2\mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2 + 2\left|A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{V'}^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left|A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{V'}^2 =_{V'} \left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), \varphi\right\rangle_V \\ = &\int_{\mathcal{O}}\left(\nabla\tilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\cdot\nabla\varphi - KJ_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial\varphi}{\partial x_3}\right)d\xi + \int_{\Gamma_u}\alpha Tr\left(\tilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right)Tr\left(\varphi\right)d\sigma,\end{aligned}$$

where φ satisfies

$$-\Delta\varphi = A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right), \text{ on } \mathcal{O}, \ \frac{\partial\varphi}{\partial\nu} + \alpha\varphi = 0, \text{ on } \Gamma_u, \ \frac{\partial\varphi}{\partial\nu} = 0, \text{ on } \Gamma_s.$$

Green's formula gives

$$\left|A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{V'}^2 = \int_{\mathcal{O}}\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\xi - \int_{\mathcal{O}}KJ_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial\varphi}{\partial x_3}d\xi =: I_1 + I_2. \tag{24}$$

By using the form of the operator A_λ, and the fact that the Tr function is bounded by the $H^1(\mathcal{O})$ norm, it follows that

$$\begin{aligned}I_1 &= \int_{\mathcal{O}}\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\xi \\ &= \int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi - \int_{\mathcal{O}}KJ_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial}{\partial x_3}\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\xi \\ &\quad + \int_{\Gamma_u}\alpha\left|Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right)\right|^2 d\sigma \\ &\leq C\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi + C\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi + \alpha_M\left|\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{H^1(\mathcal{O})}^2.\end{aligned}$$

Since $\widetilde{\Psi}_\lambda$ is assumed to be Lipschitz

$$\int_{\mathcal{O}}\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\xi \leq C\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi + C(\lambda)\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi.$$

On the other hand, using arguments in the spirit of (10),

$$I_2 = -K\int_{\mathcal{O}}J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial\varphi}{\partial x_3}d\xi \leq C\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi + \frac{1}{2}\left|A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{V'}^2.$$

By replacing I_1 and I_2 in (24), we get

$$\frac{1}{2}\left|A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|_{V'}^2 \leq C\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi + C(\lambda)\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi. \tag{25}$$

Since $\mathbb{E}\int_0^t\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi$ is bounded uniformly in ε, it is now sufficient to bound the first term on the right-hand of (25).

First, the reader is invited to note that

$$\begin{aligned}(26)\quad &\int_{\mathcal{O}} \nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\cdot\nabla J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)d\xi=\int_{\mathcal{O}} \nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\cdot\nabla\widetilde{\Psi}_\lambda^{-1}\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right)d\xi\\ &=\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2\left(\widetilde{\Psi}_\lambda^{-1}\right)'\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\xi\geq c\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi,\end{aligned}$$

because $\tilde{\Psi}_\lambda^{-1}$ is strictly increasing which follows from the fact that $\widetilde{\Psi}_\lambda$ is Lipschitz-continuous. The constant $c>0$ is generic, but we write it as a lower case since we will be employing it hereafter. The definition of A_λ yields

$$\begin{aligned}&\int_{\mathcal{O}} \nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\cdot\nabla J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)d\xi\\ \leq\ & {}_{V'}\left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_V+K\int_{\mathcal{O}} J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial}{\partial x_3}J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)d\xi\\ &-\int_{\Gamma_u}\alpha Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right)Tr\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)d\sigma\\ \leq\ & {}_{V'}\left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_V+K\int_{\mathcal{O}} J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\frac{\partial}{\partial x_3}J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)d\xi.\end{aligned}$$

The latter inequality follows from the fact that

$$Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right)Tr\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)=Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right),$$

where the right-hand operator acts on $W^{1,1}$, and its argument is non-negative due to the monotonicity of $\widetilde{\Psi}_\lambda$. The reader is reminded that $\tilde{\Psi}_\lambda^{-1}$ is $\frac{1}{\lambda}$-Lipschitz continuous. As a consequence, for $\delta>0$, one has

$$\left|J_\lambda^{\mu,\varepsilon}(x)\frac{\partial}{\partial x_3}J_\lambda^{\mu,\varepsilon(x)}\right|\leq\delta\left|\nabla J_\lambda^{\mu,\varepsilon}(x)\right|^2+\frac{1}{4\delta}\left|J_\lambda^{\mu,\varepsilon}(x)\right|^2\leq C(\lambda)\delta\left|\nabla\tilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(x)\right)\right|^2+\frac{1}{4\delta}\left|J_\lambda^{\mu,\varepsilon}(x)\right|^2.$$

Owing to (26), by picking δ small enough,

$$\begin{aligned}(27)\quad c\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi\leq & {}_{V'}\left\langle A_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right\rangle_V+C(\lambda)\int_{\mathcal{O}}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|^2 d\xi\\ &+\frac{c}{2}\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\right|^2 d\xi.\end{aligned}$$

We recall the estimate (23) in order to obtain, from (27) and (25), the estimates (18), i.e.,

$$\mathbb{E}\int_0^t\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon(r)\right)\right|_{V'}^2 dr\leq C(\lambda).$$

Step 4. (strong convergences) In order to conclude the proof of the Lemma we shall now prove that

$$X_\lambda^\varepsilon\to X_\lambda\text{ strongly in }L^\infty\left(0,T;L^2\left(\Omega;V'\right)\right).\tag{28}$$

and

$$\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)\to\eta\text{ strongly in }L^2\left(0,T;L^2\left(\Omega;L^2\left(\mathcal{O}\right)\right)\right).\tag{29}$$

To this purpose, we apply Itô's formula with the V' squared norm to $X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}$ on $[0,t]$.

$$\begin{aligned}&\mathbb{E}\left|X_\lambda^\varepsilon(t)-X_\lambda^{\varepsilon'}(t)\right|_{V'}^2+2\mathbb{E}\int_0^t\left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-A_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right),X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right\rangle_{V'}dr\\ &-2\mu\mathbb{E}\int_0^t\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2 dr+2\mathbb{E}\int_0^t\left\langle F_u+F_s,X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right\rangle_{V'}dr\leq C\mathbb{E}\int_0^t\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2 dr.\end{aligned}$$

For the term involving the differential operators $A_\lambda^{\mu,\cdot}$, one has

(30)

$$\begin{aligned}
&\left\langle A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-A_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right),X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right\rangle_{V'}\\
=&\left\langle A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right\rangle_{V'}\\
&+\left\langle A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right),\varepsilon A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\varepsilon' A_\lambda^{\mu}\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right)\right\rangle_{V'}\\
\geq&\mu\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2+\left\langle\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon}\left(X_\lambda^{\varepsilon'}\right)\right\rangle_2\\
&-K\left\langle J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right),\frac{\partial\varphi}{\partial x_3}\right\rangle_2-2\left(\varepsilon+\varepsilon'\right)\left(\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2+\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2\right)
\end{aligned}$$

where φ satisfies

$$-\Delta\varphi=J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon}\left(X_\lambda^{\varepsilon'}\right),\ \text{on }\mathcal{O},\ \frac{\partial\varphi}{\partial\nu}+\alpha\varphi=0,\ \text{on }\Gamma_u,\ \frac{\partial\varphi}{\partial\nu}=0,\ \text{on }\Gamma_s.$$

Going back to (30), one gets

$$\begin{aligned}
&\mathbb{E}\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2+2\mu\mathbb{E}\int_0^t\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2dr+2\lambda\mathbb{E}\int_0^t\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_2^2dr\\
&+2\mathbb{E}\int_0^t\left\langle\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right\rangle_2dr\\
&-2K\mathbb{E}\int_0^t\left\langle J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right),\frac{\partial\varphi}{\partial x_3}\right\rangle_2dr\\
&-4\left(\varepsilon+\varepsilon'\right)\mathbb{E}\int_0^t\left(\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2+\left|A_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2\right)dr\leq C\mathbb{E}\int_0^t\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2dr.
\end{aligned}$$

Since

$$\begin{aligned}
&-2K\left\langle J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right),\frac{\partial\varphi}{\partial x_3}\right\rangle_2\\
\geq&-\lambda\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_2^2-\frac{K^2}{\lambda}\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2,
\end{aligned}$$

by the same argument as in the Lemma 3 (see (10)), and since the strong monotonicity of Ψ_λ^{-1} yields, for some generic $c(\lambda)>0$,

$$\begin{aligned}
&\left\langle\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right),J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon}\left(X_\lambda^{\varepsilon'}\right)\right\rangle_2\\
\geq&\ c\left(\lambda\right)\left|\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right)\right|_2^2,
\end{aligned}$$

one gets

$$\begin{aligned}
&\mathbb{E}\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2+\left(2\mu-\frac{K^2}{\lambda}\right)\mathbb{E}\int_0^t\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2dr\\
&+\lambda\mathbb{E}\int_0^t\left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)-J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_2^2dr+2c(\lambda)\mathbb{E}\int_0^t\left|\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right)-\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right)\right|_2^2dr\\
&-4\left(\varepsilon+\varepsilon'\right)\mathbb{E}\int_0^t\left(\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right|_{V'}^2+\left|A_\lambda^{\mu,\varepsilon}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2\right)dr\leq C\mathbb{E}\int_0^t\left|X_\lambda^\varepsilon-X_\lambda^{\varepsilon'}\right|_{V'}^2dr.
\end{aligned}$$

Finally, by Gronwall's inequality and keeping in mind (18), for μ sufficiently large (larger than $\frac{K^2}{2\lambda}$), one gets

$$\begin{aligned}(31)\quad &\mathbb{E}\left|X_\lambda^\varepsilon - X_\lambda^{\varepsilon'}\right|_{V'}^2 + \mathbb{E}\int_0^t \left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) - J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_{V'}^2 dr + \mathbb{E}\int_0^t \left|J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right) - J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right|_2^2 dr \\ &+ \mathbb{E}\int_0^t \left|\Psi_\lambda\left(J_\lambda^{\mu,\varepsilon}\left(X_\lambda^\varepsilon\right)\right) - \Psi_\lambda\left(J_\lambda^{\mu,\varepsilon'}\left(X_\lambda^{\varepsilon'}\right)\right)\right|_2^2 dr \leq C\left(\lambda\right)\left(\varepsilon + \varepsilon'\right).\end{aligned}$$

The proof of the lemma is complete by letting $\varepsilon, \varepsilon' \to 0$. ■

4 Appendix

4.1 Estimates on the eigenfunctions and well-posedness of Σ

Let us consider the Laplace operator with Robin boundary conditions on an open bounded domain $\mathcal{O} \in \mathbb{R}^d$ with regular boundary. We consider the eigenfunctions $\{e_j\}_j$ and the eigenvalues $\{\lambda_j\}_j$, that is, we look into the following problem

$$\begin{cases} -\Delta e_j = \lambda_j e_j, & \text{on } \mathcal{O}, \\ \dfrac{\partial e_j}{\partial \nu} + \alpha e_j = 0, & \text{on } \Gamma_u, \\ \dfrac{\partial e_j}{\partial \nu} = 0, & \text{on } \Gamma_s. \end{cases}$$

where $\partial\mathcal{O} = \Gamma$ is sufficiently smooth, formed by the disjoint parts Γ_u and Γ_s, i.e.

$$\Gamma = \overline{\Gamma}_u \cup \Gamma_s, \quad \Gamma_u \cap \Gamma_s = \emptyset.$$

This problem has a sequence of solutions $(\lambda_j, \varphi_j)_{j\in\mathbb{N}}$ with $\lambda_j \geq 0$ a growing sequence which tends to infinity. This result can be easily deduced using classical results on self-adjoint operators with compact resolvant, see for instance Theorem 3.10, page 53 in [7].

Proposition 8 *There exist two real constants $\overline{C} > 0$ and $\widetilde{C} > 0$ such that, for every $x \in L^2(\mathcal{O})$,*

$$|xe_j|_2^2 \leq \widetilde{C}\lambda_j^{\frac{d-1}{2}} |x|_2^2, \quad \forall j \in \mathbb{N},$$

and

$$|xe_j|_{V'}^2 \leq \overline{C}\left(1 + \lambda_j^{\frac{d+1}{2}}\right) |x|_{V'}^2, \quad \forall j \in \mathbb{N}.$$

Proof. To prove the first assertion, one notes that

$$|xe_j|_2 \leq |e_j|_\infty |x|_2 \leq C\lambda_j^{\frac{d-1}{4}} |x|_2, \quad \forall j \in \mathbb{N}.$$

Indeed, in the spirit of of [17, Theorem 1], we have that $|e_j|_\infty \leq \widetilde{C}\lambda_j^{\frac{d-1}{4}}$ for all $k \in \mathbb{N}$. The reader is invited to note that we consider the eigenvalues to be λ_j, while [17] deals with λ_j^2 as eigenvalue. The constant C is generic and depends on the domain $\mathcal{O}$, but not on j.
In order to prove the second inequality, we consider the functional framework which was introduced at the beginning of the paper. One has

$$|xe_j|_{V'}^2 = \langle xe_j, xe_j\rangle_{V'} = (xe_j)(\varphi),$$

where φ is the solution of

$$-\Delta\varphi = e_j x, \text{ on } \mathcal{O}, \ \frac{\partial\varphi}{\partial\nu} + \alpha\varphi = 0, \text{ on } \Gamma_u, \ \frac{\partial\varphi}{\partial\nu} = 0, \text{ on } \Gamma_s.$$

Since $x \in L^2(\mathcal{O})$, it follows that

$$|(e_j x)(\varphi)| = \left|\langle e_j x, \varphi\rangle_2\right| = \left|\int_{\mathcal{O}} e_j(\xi)\, x(\xi)\, \varphi(\xi)\, d\xi\right| \leq |x|_{V'} |e_j\varphi|_V \tag{32}$$

by the Gelfand triple $V \subset L^2(\mathcal{O}) \subset V'$. One computes

$$\begin{aligned} |e_j\varphi|_V^2 &= \int_{\mathcal{O}} |\nabla(e_j\varphi)|^2 d\xi + \int_{\Gamma_u} \alpha |e_j\varphi|^2 d\sigma \\ &= -\int_{\mathcal{O}} \Delta(e_j\varphi)(e_j\varphi) d\xi + \int_{\partial\mathcal{O}} \frac{\partial(e_j\varphi)}{\partial\nu}(e_j\varphi) d\sigma + \int_{\Gamma_u} \alpha |e_j\varphi|^2 d\sigma. \end{aligned}$$

The reader is invited to note that, by a slight abuse of notations, we have dropped the Tr operator, but it should still be kept on all the elements integrated w.r.t $d\sigma$. For the first term , one gets

$$\begin{aligned} -\int_{\mathcal{O}} \Delta(e_j\varphi)(e_j\varphi) d\xi &= -\int_{\mathcal{O}} \left(e_j\varphi^2\Delta e_j + e_j^2\varphi\Delta\varphi + \frac{1}{2}\nabla\left(e_j^2\right) \cdot \nabla\left(\varphi^2\right)\right) d\xi \\ &= -\int_{\mathcal{O}} \left(e_j\varphi^2\Delta e_j + e_j^2\varphi\Delta\varphi\right) d\xi + \frac{1}{2}\int_{\mathcal{O}} e_j^2\Delta\left(\varphi^2\right) d\xi - \frac{1}{2}\int_{\partial\mathcal{O}} e_j^2\frac{\partial\varphi^2}{\partial\nu} d\sigma. \end{aligned}$$

Going back to the initial expression yields

$$\begin{aligned} |e_j\varphi|_V^2 &= -\int_{\mathcal{O}} \left(e_j\varphi^2\Delta e_j + e_j^2\varphi\Delta\varphi\right) d\xi + \frac{1}{2}\int_{\mathcal{O}} e_j^2\Delta\left(\varphi^2\right) d\xi \\ &\quad -\frac{1}{2}\int_{\partial\mathcal{O}} e_j^2\frac{\partial\varphi^2}{\partial\nu} d\sigma + \int_{\partial\mathcal{O}} \frac{\partial(e_j\varphi)}{\partial\nu}(e_j\varphi) d\sigma + \int_{\Gamma_u} \alpha |e_j\varphi|^2 d\sigma \\ &= \int_{\mathcal{O}} \lambda_j e_j^2\varphi^2 d\xi - \int_{\mathcal{O}} e_j^2\varphi\Delta\varphi d\xi + \frac{1}{2}\int_{\mathcal{O}} e_j^2\left(2\varphi\Delta\varphi + 2|\nabla\varphi|^2\right) d\xi \\ &\quad -\frac{1}{2}\int_{\mathcal{O}} e_j^2\frac{\partial\varphi^2}{\partial\nu} d\sigma + \int_{\partial\mathcal{O}} \frac{\partial(e_j\varphi)}{\partial\nu}(e_j\varphi) d\sigma + \int_{\Gamma_u} \alpha |e_j\varphi|^2 d\sigma \\ &= \int_{\mathcal{O}} \left(\lambda_j e_j^2\varphi^2 d\xi + e_j^2|\nabla\varphi|^2\right) d\xi - \frac{1}{2}\int_{\partial\mathcal{O}} e_j^2\frac{\partial\varphi^2}{\partial\nu} d\sigma \\ &\quad + \int_{\partial\mathcal{O}} \frac{\partial(e_j\varphi)}{\partial\nu}(e_j\varphi) d\sigma + \int_{\Gamma_u} \alpha |e_j\varphi|^2 d\sigma =: I + B_1 + B_2 + B_3. \end{aligned}$$

Let us write

$$B_2 = \int_{\partial\mathcal{O}} \frac{\partial(e_j\varphi)}{\partial\nu}(e_j\varphi) d\sigma = \int_{\partial\mathcal{O}} \frac{1}{2}\frac{\partial\left(e_j^2\varphi^2\right)}{\partial\nu} d\sigma = \int_{\partial\mathcal{O}} \frac{1}{2}\left(\frac{\partial\left(e_j^2\right)}{\partial\nu}\varphi^2 + \frac{\partial\left(\varphi^2\right)}{\partial\nu}e_j^2\right) d\sigma.$$

Since

$$\frac{1}{2}\frac{\partial\left(e_j^2\right)}{\partial\nu}=2e_j\frac{\partial e_j}{\partial\nu}=\left\{\begin{array}{ll}0, & \text{on } \Gamma_s,\\ -\alpha e_j^2, & \text{on } \Gamma_u,\end{array}\right.$$

it follows that

$$B_2=-\int\limits_{\partial\mathcal{O}}\alpha\left|e_j\varphi\right|^2d\sigma+\frac{1}{2}\int\limits_{\partial\mathcal{O}}\frac{\partial\left(\varphi^2\right)}{\partial\nu}e_j^2d\sigma.$$

Replacing in the previous relation yields

$$\left|e_j\varphi\right|_V^2=\int\limits_{\mathcal{O}}\left(\lambda_je_j^2\varphi^2d\xi+e_j^2\left|\nabla\varphi\right|^2\right)d\xi.$$

Then

$$\left|e_j\varphi\right|_V^2\leq C\lambda_j^{\frac{d+1}{2}}\int\limits_{\mathcal{O}}\varphi^2d\xi+C\lambda_j^{\frac{d-1}{2}}\int\limits_{\mathcal{O}}\left|\nabla\varphi\right|^2d\xi\leq C\left(1+\lambda_j^{\frac{d+1}{2}}\right)\left|\varphi\right|_V^2.$$

As a consequence,

$$\left|\varphi\right|_V^2=\left\langle e_jx,\varphi\right\rangle_2\leq\left|x\right|_{V'}\left|e_j\varphi\right|_V\leq C\sqrt{\left(1+\lambda_j^{\frac{d+1}{2}}\right)}\left|\varphi\right|_V\left|x\right|_{V'}.$$

Going back to (32), one gets, by combining the last two inequalities, the remaining assertion of our proposition. ■

4.2 Proofof Lemma 6

Proof of Lemma 6. We write the weak form of the difference

$$J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})+\varepsilon A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\varepsilon A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)=y-\overline{y},\quad in\ V',$$

as

$$_{V'}\left\langle J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y}),\zeta\right\rangle_V+\varepsilon{}_{V'}\left\langle A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}(y)\right)-A_\lambda^\mu\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right),\zeta\right\rangle_V={}_{V'}\left\langle y-\overline{y},,\zeta\right\rangle_V,$$

for $\zeta\in V$ and we take the particular element $\zeta=\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\in V$ to get

$$\begin{aligned}
&{}_{V'}\left\langle J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y}),\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right\rangle_V\\
&+\varepsilon\mu\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)d\xi\\
&+\varepsilon\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right|^2d\xi\\
&-\varepsilon\int_{\mathcal{O}}K\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\left(\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)\cdot\ i_3d\xi\\
&+\varepsilon\int_{\Gamma_u}\alpha Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)^2d\sigma=\int_{\mathcal{O}}(y-\overline{y})\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)d\xi.
\end{aligned}$$

From the λ-strong monotonicity of $\widetilde{\Psi}_\lambda$, we have that

$$\begin{aligned}
&(\lambda+\varepsilon\mu\lambda)\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)^2d\xi+\varepsilon\int_{\mathcal{O}}\left|\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right|^2d\xi\\
&-\varepsilon K\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\left(\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\nabla\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)\cdot\ i_3d\xi\\
&+\varepsilon\int_{\Gamma_u}\alpha Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)^2d\sigma\leq\int_{\mathcal{O}}(y-\overline{y})\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)d\xi.
\end{aligned}$$

Elementary computations yield

$$\begin{aligned}
&\left(\lambda+\varepsilon\mu\lambda-\varepsilon K^2\right)\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)^2 d\xi+\varepsilon\int_{\Gamma_u}\alpha Tr\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)^2 d\sigma\\
\leq\ &\int_{\mathcal{O}}(y-\overline{y})\left(\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(y)\right)-\widetilde{\Psi}_\lambda\left(J_\lambda^{\mu,\varepsilon}(\overline{y})\right)\right)d\xi\\
\leq\ &\frac{1}{2\varepsilon}\int_{\mathcal{O}}(y-\overline{y})^2 d\xi+\frac{\varepsilon}{2\lambda^2}\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)^2 d\xi.
\end{aligned}$$

and, therefore,

$$\left(\lambda+\varepsilon\mu\lambda-\varepsilon K^2-\frac{\varepsilon}{2\lambda^2}\right)\int_{\mathcal{O}}\left(J_\lambda^{\mu,\varepsilon}(y)-J_\lambda^{\mu,\varepsilon}(\overline{y})\right)^2 d\xi\leq\frac{1}{2}\int_{\mathcal{O}}(y-\overline{y})^2 d\xi.$$

For μ sufficiently large e.g., larger than $\frac{K^2+1}{\lambda^3}$, we have the Lipschitz continuity of $J_\lambda^{\mu,\varepsilon}$ for every $0<\lambda,\varepsilon<1$. Note that, alternatively, the conclusion can be obtained with a fixed μ by requiring ε to be small enough. ■

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