A phase transition in block-weighted random maps

William Fleurat, Zéphyr Salvy

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Abstract

We consider the model of random planar maps of size n biased by a weight $u > 0$ per 2-connected block, and the closely related model of random planar quadrangulations of size n biased by a weight $u > 0$ per simple component. We exhibit a phase transition at the critical value $u_C = 9/5$. If $u < u_C$, a condensation phenomenon occurs: the largest block is of size $\Theta(n)$. Moreover, for quadrangulations we show that the diameter is of order $n^{1/4}$, and the scaling limit is the Brownian sphere. When $u > u_C$, the largest block is of size $\Theta(\log(n))$, the scaling order for distances is $n^{1/2}$, and the scaling limit is the Brownian tree. Finally, for $u = u_C$, the largest block is of size $\Theta(n^{2/3})$, the scaling order for distances is $n^{1/3}$, and the scaling limit is the stable tree of parameter $3/2$.

1 Introduction

Models of planar maps exhibit a form of *universality*: many "natural" classes of random maps exhibit a similar behaviour when the size grows to infinity. This can be made precise by considering *scaling limits*: when taking an object $\mathbf{M}_n$ uniformly among all objects of size n in some class, then, after an appropriate rescaling, the sequence $(\mathbf{M}_n)_{n\geqslant 1}$ converges in distribution towards some random metric space. This was first proved for uniform quandrangulations by Miermont [Mie13] and independently for the cases of uniform $2q$-angulations ($q \geqslant 2$) and uniform triangulations by Le Gall [LG13], following a sequence of results on this subject [MM03, CS04, LG07, LG10, LGM11]. Since then, these results have been extended to other families of maps: the sequence $(\mathbf{M}_n)$ converges towards the *Brownian sphere* $\mathcal{M}_e$ (also called Brownian map, see Fig. 1), always with a rescaling by $cn^{1/4}$ for some model-dependent $c > 0$. Gromov-Hausdorff's topology allows to make sense of the convergence of a sequence of maps to a certain limit, considering them as (isometry classes of) compact metric spaces. In particular, uniform planar maps also converge towards the Brownian sphere [BJM14], as well as other families such as uniform triangulations and uniform $2q$-angulations ($q \geqslant 2$) [LG13], uniform simple triangulations and uniform simple quadrangulations [ABA17], bipartite planar maps with a prescribed face-degree sequence [Mar18], $(2q+1)$-angulations [ABA21] and Eulerian triangulations [Car21].

On the other hand, "degenerate" classes of maps that "look like" trees exhibit another universality phenomenon. In particular, upon rescaling by $cn^{1/2}$, there is a convergence to the *Brownian tree* $\mathcal{T}^{(2)}$ (see Fig. 2), the scaling limit of critical Galton-Watson trees with finite variance [Ald93, LG06]. This is the case for classes of maps with a tree-decomposition such as stack triangulations [AM08];

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Figure 1: Approximation of the Brownian sphere by a simple quadrangulation of size 50 000, using a generator by Éric Fusy.

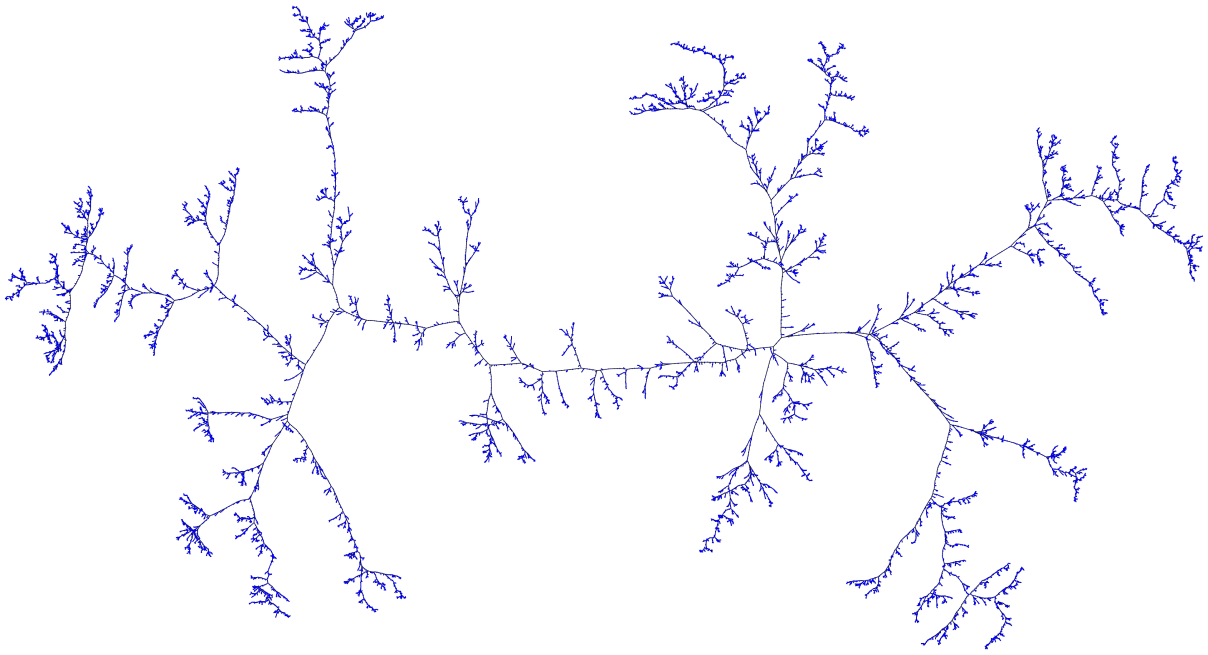

Figure 2: Approximation of the Brownian tree by a binary tree of size approximately 70 000.

classes of maps with some particular boundary conditions, such as quadrangulations of a polygon [Bet15], outerplanar maps [Car16]; or, more generally for "subcritical" classes [Stu20] (see [PSW16] for the case of graphs).

Models interpolating between the Brownian tree and the circle can be obtained by using *looptrees* [CK13b]. Curien and Kortchemski considered the boundary of large percolation clusters in the *uniform infinite planar triangulation* (which is the local limit of large triangulations) where each vertex is coloured (independently) white with probability $a \in (0,1)$ and black otherwise. They showed that if $a \in (0,1/2)$, the scaling limit is the Brownian tree, if $a \in (1/2,1)$ it is the unit circle and if $a = 1/2$ it is the *stable looptree of parameter* $3/2$ [CK13a], which correspond to the *stable tree of parameter* $3/2$ (see Fig. 3) where each branching point is replaced by a circle. Richier [Ric18] also showed that the boundary of critical Boltzmann planar maps with face-degrees in the domain of attraction of a stable distribution with parameter $\alpha \in (1,2]$ exhibit a similar phase transition: if $\alpha \in (1,3/2)$, the scaling limit is the stable looptree of parameter $(\alpha - 1/2)^{-1}$, and, with Kortchemski, Richier showed that it is the circle of unit length if $\alpha \in (3/2,2]$ and conjectured that this holds also for $\alpha = 3/2$ [KR20]. Stefánsson and Stufler showed that face-weighted outerplanar maps have a similar phase diagram, with looptrees the α-stable looptree being the scaling limit when their $\alpha \in (1,2)$, the Brownian tree when $\alpha = 2$ and the deterministic circle of unit length when $\alpha = 1$ [SS19]. In all three cases, the parameter of the model allows the number of cut vertices appearing on the boundary to be adjusted, thus changing from a "round" to a "tree" phase.

Some natural models also interpolate between the Brownian sphere and the Brownian tree. For example, consider random quadrangulations with n faces and a boundary of length ℓ, where $\ell/\sqrt{n} \to \sigma$. When $\sigma = 0$, the scaling limit is the Brownian sphere, when $\sigma = \infty$ it is the Brownian

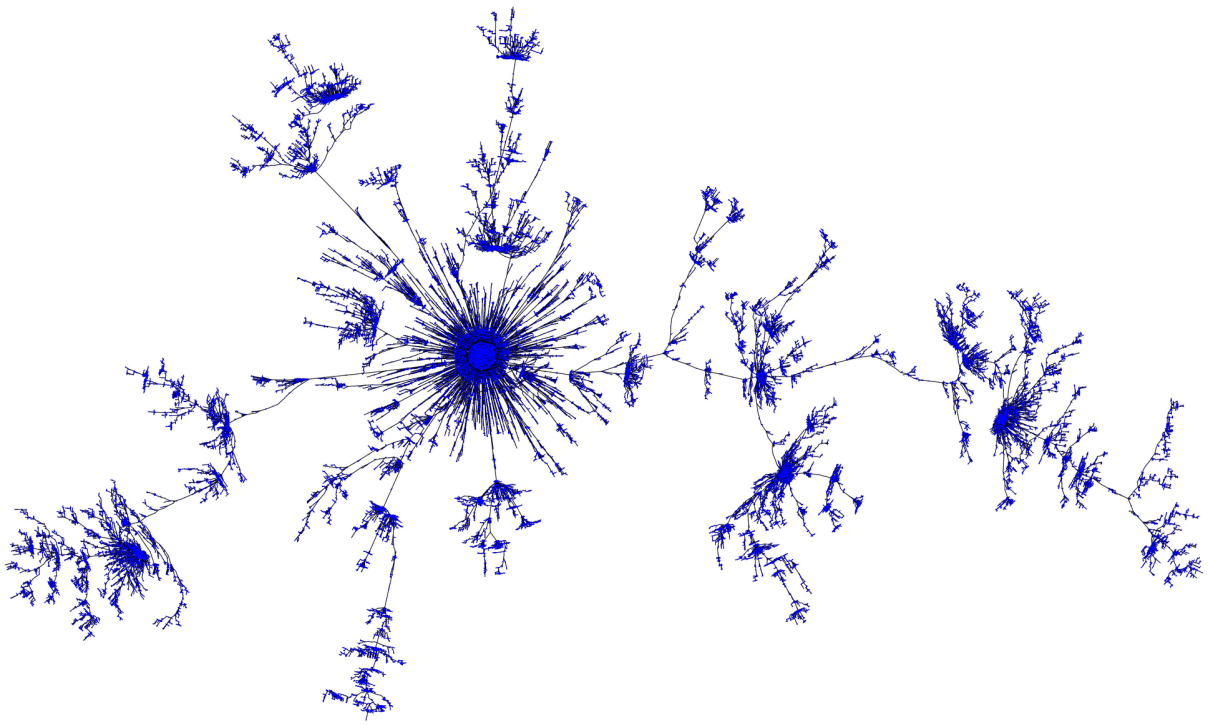

Figure 3: Approximation of the stable tree 3/2 by a tree of size approximately 150 000.

tree, and for all $\sigma \in (0,\infty)$ it is the *Brownian disk* with boundary length σ [Bet15]. Another example is random bipartite planar maps with properly normalized face-weights, which converge towards the Brownian tree when the distribution for the weights has expected value smaller than 1 [JS15], and towards the Brownian map when the expected value is 1 and the variance is finite [Mar18]. Moreover, when the the expected value is 1 and the distribution is in the domain attraction of a stable law of parameter $\alpha \in (1,2)$, these maps converge, at least along suitable subsequences, towards a limit which is not the Brownian sphere, and is conjectured to be the *stable map* of parameter α [LGM11].

Model. The purpose of this paper is to propose yet another model interpolating between the Brownian sphere and the Brownian tree, but where the transition does not appear through the boundary. It relies on a parameter tuning the density of separating elements. In this model, a map $\mathfrak{m}$ is sampled with a probability that depends on its number $b(\mathfrak{m})$ of maximal 2-connected components, or "blocks", for which a precise definition will be given later in Section 2.

In fact, we will consider two probability distributions on maps, both indexed by a parameter $u > 0$. The first one is a fixed size model: for any $n \in \mathbb{Z}_{\geqslant 0}$, we define

$$\mathbb{P}_{n,u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n]M(z,u)} \quad \text{for any } \mathfrak{m} \in \mathcal{M}_n, \tag{1}$$

where $\mathcal{M}_n$ is the set of maps with n edges, $M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} u^{b(\mathfrak{m})} z^{|\mathfrak{m}|} = \sum_{n\in\mathbb{Z}_{\geqslant 0}} ([z^n]M(z,u))\, z^n$ and $|\mathfrak{m}|$ is the number of edges of $\mathfrak{m}$. The second one is a Boltzmann-type distribution which samples maps with random sizes. More precisely, write $\rho(u)$ for the radius of convergence of $z \mapsto M(z,u)$. We define[1]:

$$\mathbb{P}_u(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{M(\rho(u),u)}\rho(u)^{|\mathfrak{m}|} \quad \text{for any } \mathfrak{m} \in \mathcal{M}. \tag{2}$$

The qualitative properties of maps sampled according to these measures change drastically when u varies, and we will see that it gives rise to different regimes with a phase transition. Examples of such maps are represented on Figs. 4 to 8. In this paper, blocks will be either maximal 2-connected components of maps, or maximal simple components of quadrangulations. Indeed, both models have the same underlying structure, so one study gives results for both (see Sections 2.4 and 2.5), except for some of the scaling limit results, where some convergence results for 2-connected maps are missing. However, our approach could be generalised to many other models with an underlying tree structure (see Table 3), such as the ones described in [BFSS01]. In particular, the case $u = 1$ corresponds to sampling a uniform map and $u \to 0$ to sampling a uniform block.

Block decompositions have already been used in the context of scaling limits, and some joint convergences are known: a quadrangulation, its largest 2-connected block, and its largest simple block jointly converge to the same Brownian sphere [ABW17].

The scaling limit of a tree-decomposed model like ours depends on the geometries of the blocks and of the underlying decomposition tree. In our setting, one of the behaviour always ends up dominating, but this is not always the case: Sénizergues, Stefánsson and Stufler study situations where both geometries play a role in the scaling limit, and define the *decorated α-stable trees* which are the corresponding scaling limits [SSS22]. Our results for the scaling limits in the critical and supercritical cases confirm their conjecture in [SSS22, Remark 1.1]. They build on a model introduced by Archer, which, contrary to this work, develops the *local limit* point of view [Arc20, Chapter 6]. In

[1]The finiteness of $M(\rho(u),u)$ is justified in Section 2.2.

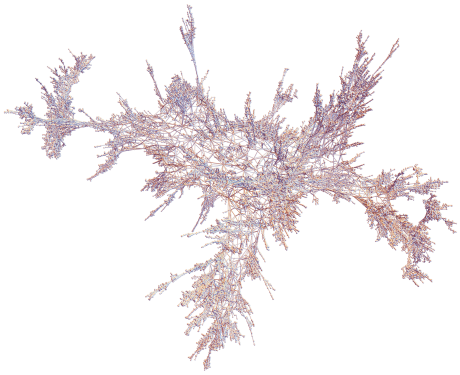

Figure 4: Map drawn according to the subcritical model $\mathbb{P}_{n,1}$ of size around 55 000 (see larger version in Fig. 20).

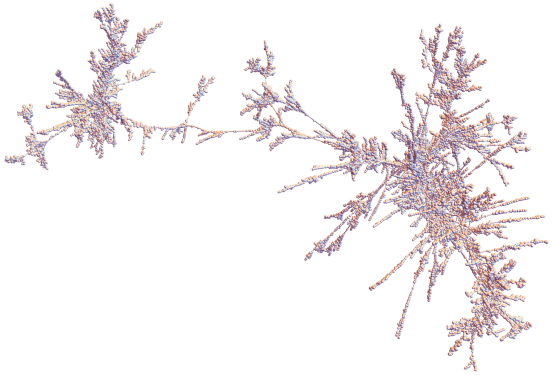

Figure 5: Map drawn according to the subcritical model $\mathbb{P}_{n,8/5}$ of size around 55 000 (see larger version in Fig. 21).

particular, Archer shows that the fractal dimension of the local limit for the critical and supercritical cases are respectively 3 and 2[2]. Both cases correspond to what Archer called the "tree regime", where the local geometry of the tree is preponderant in the limit. Both articles consider only *critical* offspring distributions for the trees, which does not hold in our subcritical regime.

The model with a weight per 2-connected blocks was already analysed with a combinatorics point of view by Bonzom [Bon16, §8] with physical applications in mind (see [DS09] for a thorough discussion). The so-called *quadric* model studied in his work can be specialized to our model. Bonzom obtains rational parametrisations for the generating series, and exhibits the possible singular behaviours, which suggest the existence of three different regimes: a "map behaviour", a "tree-behaviour", and in-between a "proliferation of baby universes". Since his focus is much broader, he does not go into details to study this particular model from a probabilistic point of view, and this is the main topic of the present article. For $u = 1$, which corresponds to sampling maps uniformly, this model has also been studied with the point of view of block decomposition in [BFSS01] and [AB19].

Results. Our results are summarized in Table 1. In Section 4, we show that, with high probability, when $u < 9/5$, there is condensation with one block of size $\Theta(n)$ and all others of size $O(n^{2/3})$, see Theorem 2; when $u > 9/5$, the largest block has size $\Theta(\log(n))$, see Theorem 3; and when $u = 9/5$ the largest block is of size $\Theta(n^{2/3})$, see Theorem 4.

In Section 5, we give a unified proof of the convergence towards $\mathcal{T}^{(2)}$, after renormalising distances by $n^{1/2}$, in the supercritical case $u > 9/5$; and towards $\mathcal{T}^{(3/2)}$, after renormalising distances by $n^{2/3}$, in the critical case $u = u_C$ (Theorem 5). For $u > 9/5$, we retrieve a previous result by Stufler for more general weighted models [Stu20]. All these results hold for both maps and their 2-connected cores, and quadrangulations and their simple cores. Finally, when $u < 9/5$, we show in Theorem 6 that quadrangulations converge towards the Brownian sphere when renormalising distances by $n^{1/4}$. In the case of quadrangulations, these results are consistent with existing literature for the case $u = 1$ [Mie13, LG13, BJM14], as well as when $u \to 0$ [ABA17]. We rely crucially on the convergence of uniform *simple* quadrangulations with the same normalisation, which is proven in [ABA17],

[2]This uses that the diameter for uniform blocks is $\Theta\left(n^{1/4}\right)$, which is known for simple quadrangulations but only assumed for 2-connected maps.

Figure 6: Map drawn according to the critical model $\mathbb{P}_{n,9/5}$ of size around 80 000 (see larger version in Fig. 22).

Figure 7: Map drawn according to the supercritical model $\mathbb{P}_{n,5/2}$ of size around 75 000 (see larger version in Fig. 23).

Figure 8: Map drawn according to the supercritical model $\mathbb{P}_{n,5}$ of size around 50 000 (see larger version in Fig. 24).

	Largest block	Scaling	Scaling limit
$u < 9/5$	$\Theta(n)$	$n^{1/4}$	Brownian sphere[3] $\mathcal{M}_e$
$u = 9/5$	$\Theta(n^{2/3})$	$n^{1/3}$	Stable tree $\mathcal{T}^{(3/2)}$
$u > 9/5$	$\Theta(\log(n))$	$n^{1/2}$	Brownian tree $\mathcal{T}^{(2)}$

Table 1: Behaviour of the model when u varies.

and recalled in Proposition 24 below. A similar convergence result for uniform 2-connected maps would be needed in order to prove a version of Theorem 6 for maps, see the discussion after the statement of Proposition 24. Such a convergence is expected to hold and hinted at for instance by Lehéricy's results [Leh22], which show that graph distances on a uniform map of size n and on its quadrangulation via Tutte's bijection behave similarly when $n \to \infty$.

Section 2 and Theorem 1 introduce tools to prove these theorems. We show that maps and quadrangulations can be decomposed into blocks with an underlying tree structure. We show that the law of such trees can be described by a Galton-Watson model (as in several papers cited above). From there, we exhibit in Section 3 a phase diagram going from a condensation phenomenon ($u < 9/5$) to a critical "generic" regime ($u > 9/5$) going through a "non-generic" critical point ($u = 9/5$).

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2 Tree decomposition of maps

2.1 Maps and their enumeration

A *planar map* $\mathfrak{m}$ is the proper embedding into the two-dimensional sphere of a connected planar finite multigraph, considered up to homeomorphisms. Let $V(\mathfrak{m})$ be the set of its vertices, $E(\mathfrak{m})$ the set of its edges and $F(\mathfrak{m})$ the set of its faces. The size of a planar map $\mathfrak{m}$ — denoted by $|\mathfrak{m}|$ — is defined as its number of edges.

A *half-edge* e is an oriented edge from u to v (with possibly $v = u$) and is represented as half of an edge starting from u. Its *starting vertex* u is denoted by e^- and its *end vertex* v is denoted e^+. Let $\overrightarrow{E}(\mathfrak{m})$ be the set of half-edges of $\mathfrak{m}$.

A *corner* is the angular sector between two consecutive edges in counterclockwise order around a vertex. Each half-edge is canonically associated to the corner that follows it in counterclockwise order around its starting vertex. The *degree* of a face is the number of corners incident to it.

All the maps considered in this paper are *rooted*, meaning that one of their half-edges (or one of their corners) is distinguished. The set of rooted planar maps — simply called maps in the following — is denoted by $\mathcal{M}$. For n in $\mathbb{Z}_{\geqslant 0}$, let m_n be the number of maps of size n and $M(z) = \sum_{n \in \mathbb{Z}_{\geqslant 0}} m_n z^n$ be the associated generating series. By convention, we set $m_0 = 1$ which

[3] We only prove convergence to the Brownian sphere in the case of quadrangulations and their simple blocks, see the discussion after Proposition 24.

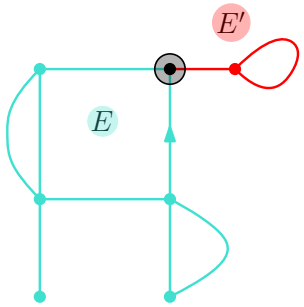

Figure 9: Example of a separable map. The circled black dot is a cut vertex.

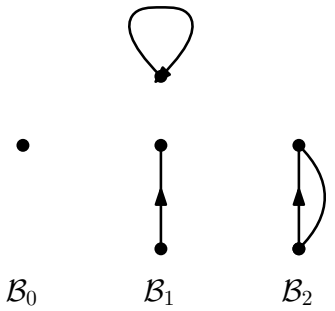

Figure 10: The classes $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ of 2-connected maps with respectively 0, 1 and 2 edges.

corresponds to the *vertex map*: the map reduced to a single vertex. Similarly, define the *edge map* as the map reduced to a single edge between two vertices.

Rooting simplifies the study by avoiding symmetry problems, however we expect our results remain true in the non-rooted setting due to the general results of [RW95]. The enumerative study of rooted planar maps was initiated by Tutte in the 60s. In particular, he obtained the following result:

Proposition 1 ([Tut63]). *The number m_n of maps of size n is equal to*

$$m_n = \frac{2(2n)!3^n}{(n+2)!n!} \sim \frac{2}{\sqrt{\pi}} 12^n n^{-5/2}, \quad n \to \infty. \tag{3}$$

This implies in particular that $\rho_M = 1/12$ and $M(\rho_M) < \infty$, where ρ_M denotes the radius of convergence of $M(z)$.

2.2 2-connected maps and block decomposition

Definition 1. *A map $\mathfrak{m} \in \mathcal{M}$ is said to be* separable *if it is possible to partition its edge-set $E(\mathfrak{m})$ into two non-empty sets E and E' such that there is exactly one vertex (called* cut vertex*) incident to both a member of E and a member of E'. The map $\mathfrak{m}$ is said to be* 2-connected *otherwise, see Fig. 9.*

Note that, by definition, the vertex map is 2-connected. For $n \in \mathbb{Z}_{\geqslant 0}$, we write $\mathcal{B}_n$ for the set of 2-connected maps of size n, and $b_n = |\mathcal{B}_n|$. From Fig. 10, we see that $b_0 = 1$, $b_1 = 2$ and $b_2 = 1$. Contrary to Tutte [Tut63], we choose $m_0 = b_0 = 1$ (and not $m_0 = b_0 = 0$) and express the results accordingly. Notice in particular that the only 2-connected map with a loop is the map reduced to a loop-edge.

Definition 2. *A* block *of a planar map $\mathfrak{m}$ is a maximal 2-connected submap* of positive size. *The number of blocks of $\mathfrak{m}$ is denoted by $b(\mathfrak{m})$.*

In the 60's, Tutte introduced the so-called "block decomposition of maps" [Tut63], which roughly speaking corresponds to cutting the map at all cut-vertices, and is illustrated on Fig. 11 (this is known for graphs as well and called *block-cut tree*, see *e.g.* [Har69]).

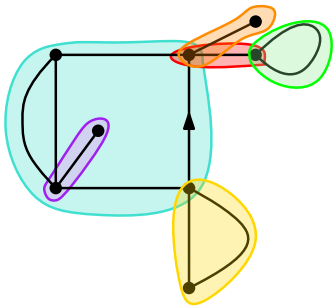

Figure 11: Decomposition of a map into blocks.

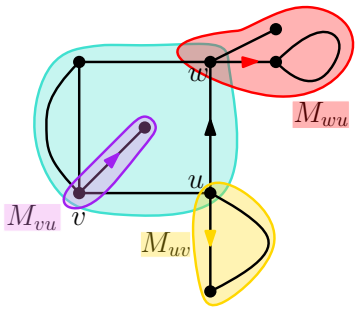

Figure 12: Pendant submap: the block to which the half-edges uv, vu and wu belong is in blue.

We describe here this decomposition drawing inspiration from Addario-Berry's presentation [AB19, §2]. Let $\mathfrak{m}$ be a map and let $\mathfrak{b}$ be the block containing its root. For each half-edge e of $\mathfrak{b}$, we define the *pendant submap* $\mathfrak{m}_e$ of e as the maximal submap of $\mathfrak{m}$ disjoint from $\mathfrak{b}$ except at e^- and located to the left of e (it is possibly reduced to the vertex map). If $\mathfrak{m}_e$ has at least one edge, we root it at the half-edge of $\mathfrak{m}$ following e in counterclockwise order around e^- (see Fig. 12).

From $\mathfrak{b}$ and the $2|E(\mathfrak{b})|$ pendant submaps $\{\mathfrak{m}_e, e \in \overrightarrow{E}(\mathfrak{b})\}$, it is possible to reconstruct the map $\mathfrak{m}$: for each $\mathfrak{m}_e$ rooted at the half-edge ρ, insert $\mathfrak{m}_e$ in the corner associated to e in such a way that ρ is the first edge after e in counterclockwise order and merge ρ^- and e^-. Thus, a map can be encoded as a block where each edge is decorated by two maps. This decomposition induces an identity of generating series, thanks to the symbolic method [FS09, Ch1]. Letting $B(y) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} b_n y^n$, Tutte's block decomposition translates into the following equality of generating series:

$$M(z) = B(zM(z)^2). \tag{4}$$

Thanks to (4) and an explicit expansion for $M(z)$ obtained in [Tut62], Tutte obtained the following enumerative results for 2-connected maps.

Proposition 2 ([Tut63]). *The number b_n of 2-connected maps of size n is*

$$b_0 = 1, \qquad \text{and for } n \geqslant 1,\ b_n = \frac{2(3n-3)!}{n!(2n-1)!} \sim \sqrt{\frac{3}{\pi}}\frac{2}{27}\left(\frac{27}{4}\right)^n n^{-5/2}, \quad n \to \infty. \tag{5}$$

Moreover, writing ρ_B for the radius of convergence of the series B, we have

$$\rho_B = \frac{4}{27}, \qquad B(\rho_B) = 4/3 \qquad \text{and} \qquad \rho_B \cdot B'(\rho_B) = \sum_{n\in\mathbb{Z}_{\geqslant 0}} n b_n \rho_B^n = 4/9. \tag{6}$$

In the following, we consider maps enumerated by both their number of edges and their numbers of blocks. Namely, we consider the following bivariate series: $M(z,u) = \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|}u^{b(\mathfrak{m})}$ (recall that $b(\mathfrak{m})$ is the number of blocks of $\mathfrak{m}$ and $|\mathfrak{m}|$ is its number of edges). Tutte's decomposition of a map into blocks translates in the following refined version of (4):

$$M(z,u) - 1 = u\left(B(zM(z,u)^2) - 1\right) \qquad \textit{i.e.} \qquad M(z,u) = uB(zM(z,u)^2) + 1 - u, \tag{7}$$

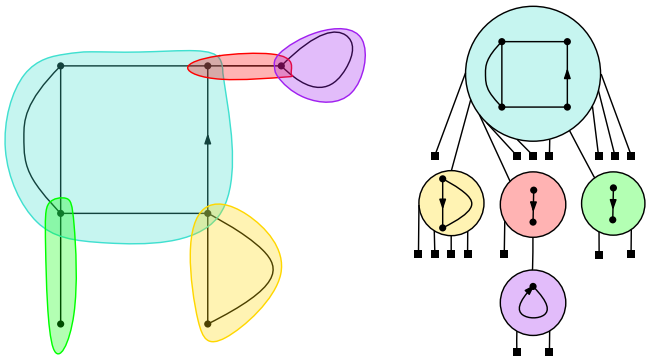

Figure 13: Block tree corresponding to a planar map.

where the term $1-u$ accounts for the fact that the vertex map has no block by Definition 2 (even if it is 2-connected). For $u>0$, denote by $\rho(u)$ the radius of convergence of $z \mapsto M(z,u)$. Since for $z \geqslant 0$ and $u \geqslant 1$

$$M(z,u) \leqslant \sum_{\mathfrak{m}\in\mathcal{M}} z^{|\mathfrak{m}|}u^{|\mathfrak{m}|} = M(uz),$$

if $|uz| < \rho_M = 1/12$, then $M(z,u)$ is a converging sum. Hence, for $u \geqslant 1$, $\rho(u) \geqslant \frac{1}{12u} > 0$. On the other hand, since $\rho(u)$ is decreasing, for $u \leqslant 1$ we have $\rho(u) \geqslant \rho(1) = \rho_M = 1/12$ (and $\rho(u) \leqslant \rho(0) = \rho_B = 4/27$).

In view of the form of the equation (7) and in particular that it is non-linear, it holds that $M(\rho(u),u) < \infty$. Indeed, since $B(y) \geqslant 1+2y$ for all $y \geqslant 0$, we get $M(z,u) \geqslant 1+2uzM(z,u)^2$. This shows that it is impossible that $M(z,u) \xrightarrow[z\to\rho(u)^-]{} +\infty$.

2.3 Block tree of a map and its applications

Tutte's block decomposition can also be applied recursively, *i.e.* we consider first the root block and then apply the block decomposition to each of the pendant submaps. By doing so, for any map $\mathfrak{m}$ we can obtain a decomposition tree $T_{\mathfrak{m}}$, which was first explicity described by Addario-Berry in [AB19, §2]. More precisely:

1. Let $\mathfrak{b} = (\mathfrak{b}, \rho)$ be the maximal 2-connected submap containing the root ρ. The root v_ρ of $T_{\mathfrak{m}}$ represents $\mathfrak{b}$, and has $2|E(\mathfrak{b})|$ children (in particular, if $\mathfrak{b}$ is of size 0, v_ρ is a leaf);

2. List the half-edges of $\mathfrak{b}$ as $a_1, \dots, a_{2|E(\mathfrak{b})|}$ according to an arbitrarily fixed deterministic order on half-edges (*e.g.* the order in a left-to-right depth first search). Let $\mathfrak{m}_i$ be the pendant submap in the corner corresponding to the half-edge a_i in $\mathfrak{b}$. The i-th pendant subtree of $T_{\mathfrak{m}}$ is the subtree encoding $T_{\mathfrak{m}_i}$.

An example of such a correspondence is described in Fig. 13. This decomposition has three essential properties, that follow directly from its definition, and that we summarize in the following proposition.

Proposition 3 ([Tut63, AB19]). *The block tree $T_{\mathfrak{m}}$ of a map $\mathfrak{m}$ satisfies the following properties:*

- *The edges of $T_{\mathfrak{m}}$ correspond to the half-edges of $\mathfrak{m}$;*
- *The internal nodes of $T_{\mathfrak{m}}$ correspond to the blocks of $\mathfrak{m}$: if an internal node v of $T_{\mathfrak{m}}$ has r children, then the corresponding block $\mathfrak{b}_v$ of $\mathfrak{m}$ has size $r/2$;*
- *The map $\mathfrak{m}$ is entirely determined by $(T_{\mathfrak{m}}, (\mathfrak{b}_v, v \in T_{\mathfrak{m}}))$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$ if v is an internal node; else, by convention, $\mathfrak{b}_v$ is the vertex map.*

By abuse of language, we might refer to $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ as the family of blocks (even if blocks necessarily have positive size). A direct consequence of this proposition is that to study the block sizes of a map $\mathfrak{m}$, it is sufficient to study the degree distribution of $T_{\mathfrak{m}}$. This is precisely the strategy developed by Addario-Berry in [AB19]. This allows him to study the block sizes of a uniform random map $\mathbf{M}_n$ of size n, by describing $T_{\mathbf{M}_n}$ as a Galton-Watson tree with an explicit degree distribution conditioned to have $2n$ edges, and one of our contributions is to extend his result to our model.

2.4 Block tree of a quadrangulation

We describe in this section how a quadrangulation can be decomposed into maximum simple quadrangular components, in the same way that a map can be decomposed into maximum 2-connected components.

Definition 3. *A* quadrangulation *is a map with all faces of degree* 4*.*

Planar quadrangulations are *bipartite*, *i.e.* their vertices can be properly bicolored in black and white. In the following, we always assume that they are endowed with the unique such coloring having a black root vertex. Although quadrangulations are maps, when an object is explicitly defined as a quadrangulation, its size will be its number of faces. Thus, a quadrangulation of size n has $2n$ edges.

Definition 4. *A* quadrangulation of the 2-gon *is a map where the* root face *— the face containing the corner associated to the root — has degree* 2 *and all other faces have degree* 4*.*

A quadrangulation of the 2-gon with at least two faces can be identified with a quadrangulation of the sphere by simply gluing together both edges of the root face.

Definition 5. *A quadrangulation is called* simple *if it has neither loops nor multiple edges.*

Definition 6. *Let e_1e_2 be a 2-cycle of a quadrangulation $\mathfrak{q}$, its* interior *is the submap of $\mathfrak{q}$ between e_1 and e_2 (both included) which does not contain the root corner of $\mathfrak{q}$. A 2-cycle is* maximal *when it does not belong to the interior of another 2-cycle.*

Definition 7. *Let e_1e_2 be a maximal 2-cycle of a quadrangulation $\mathfrak{q}$, its* pendant subquadrangulation *is defined as its interior, which is turned into a quadrangulation of the 2-gon by rooting it at the corner incident to the unique black vertex of e_1e_2.*

Let e be a half-edge of a quadrangulation $\mathfrak{q}$. If e is oriented from black to white and there exists a half-edge f such that ef is a maximal 2-cycle of $\mathfrak{q}$, then the pendant subquadrangulation *of e is the pendant subquadrangulation of ef. Else, it is the edge map (which is also a quadrangulation of the 2-gon).*

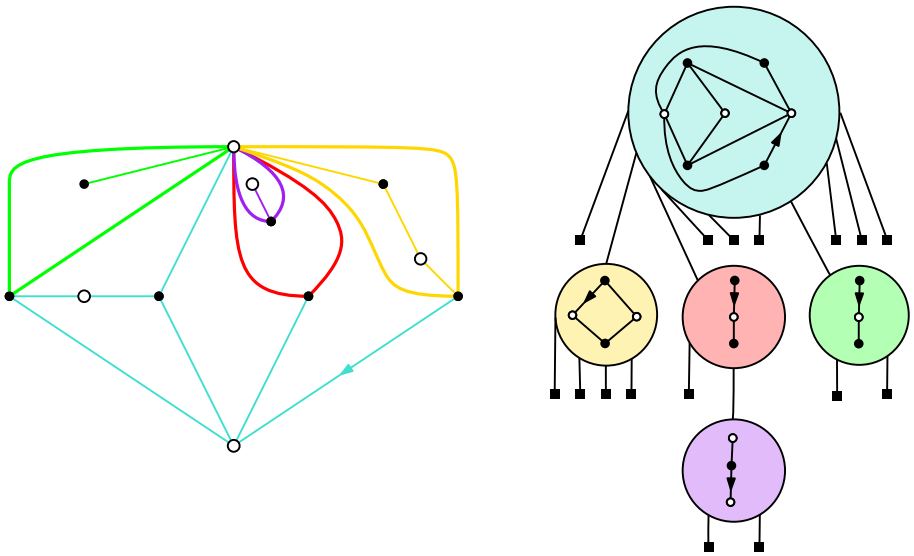

Figure 14: The image of the map of Fig. 13 via Tutte's bijection, and its block tree.

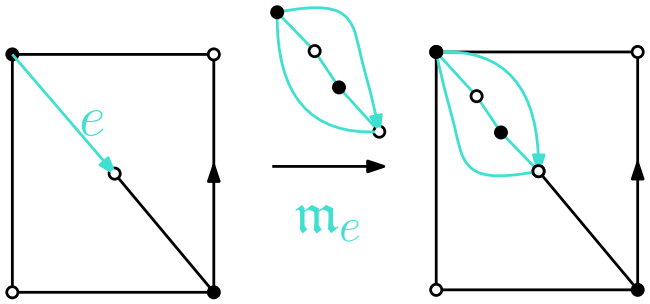

Figure 15: Reconstructing a quadrangulation from its simple core and the pendant subquadrangulations

For $\mathfrak{q}$ a quadrangulation, its *simple core* $\mathfrak{q}_s$ — the simple block containing the root — is obtained by collapsing the interior of every maximal 2-cycle of $\mathfrak{q}$. Similarly as for maps, a decomposition tree $T_{\mathfrak{q}}^{(q)}$ can be associated to a quadrangulation $\mathfrak{q}$, by recursively decomposing the pendant subquadrangulations at the simple core, see Fig. 14. *Simple blocks* are recursively defined as the simple cores appearing in the underlying arborescent decomposition. We then have an exact parallel with the situation of maps and their 2-connected components.

Given a simple quadrangulation $\mathfrak{q}_s$ and a collection of $|E(\mathfrak{q}_s)| = 2|q_s|$ quadrangulations of the 2-gon $\{\mathfrak{m}_e, e \in E(\mathfrak{q}_s)\}$, it is possible to construct a quadrangulation: for each $\mathfrak{m}_e$ of root ρ_e replace e by $\mathfrak{m}_e$ such that ρ_e has the orientation e. See Fig. 15 for an illustration. This transformation is invertible. Thus, a quadrangulation can be encoded as a simple quadrangulation where each edge is decorated by one quadrangulation of the 2-gon, *i.e.* each face is decorated by two quadrangulations of the 2-gon:

$$Q(z,u) + 1 = uS(z(Q(z,u)+1)^2) + 1 - u, \tag{8}$$

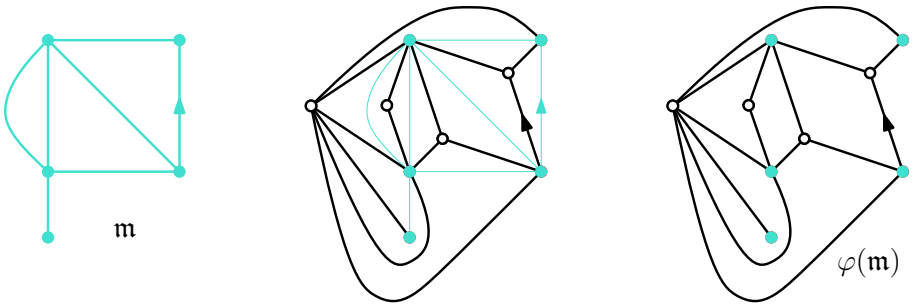

Figure 16: The quadrangulation corresponding to a map via Tutte's bijection.

where Q is the generating series for quadrangulations (with a weight z for faces, and u for simple blocks) and S is the generating series for simple quadrangulations (with a weight z for faces). Note that this equation is isomorphic to (7).

This decomposition and the former one presented for general maps are in fact two sides of the same coin. Indeed, they can be related via Tutte's bijection as we now present: there exists an explicit bijective construction between quadrangulations of size n and (general) maps of size n. More precisely, for a map $\mathfrak{m}$ (rooted in ρ), its image by φ, called its *angular map*, can be constructed as follows, see Fig. 16.

1. Add a (white) vertex inside each face of $\mathfrak{m}$ and draw an edge from this new (white) vertex to each corner around the face (respecting the order of the corners);
2. The half-edge e created in the corner of ρ is now the root, oriented from black to white;
3. Remove the original edges.

Proposition 4. *For $n \in \mathbb{Z}_{>0}$, the function φ is a bijection between maps of size n and quadrangulations of size n. Moreover, it maps bijectively 2-connected maps of size $n \geqslant 1$ to simple quadrangulations of size n.*

The construction φ is due to Tutte [Tut63, §5] (he defines the notion of derived map, from which the angular map is extracted by deleting one of the 3 classes of vertices, as explained in [Bro65, §7]). The specialization to 2-connected maps is explained *e.g.* in [Bro65]. In particular, it implies that $S(y) = B(y)$. Moreover, given Equations (7) and (8), this gives $M(z, u) = Q(z, u) + 1$.

Finally, when constructing the decomposition tree $T_{\mathfrak{q}}^{(q)}$, if the deterministic orders used for the half-edges of 2-connected maps and for the edges of simple quadrangulations are consistent via Tutte's bijection, then the decomposition trees of $\mathfrak{m}$ and of $\varphi(\mathfrak{m})$ are the same, and for each node v of the tree, the 2-connected map (resp. simple quadrangulation) at v are in correspondence by Tutte's bijection, *e.g.* the example of Fig. 13 is consistent with the example of Fig. 14 via Tutte's bijection. This can be rephrased as the following result.

Proposition 5. *For all $\mathfrak{m} \in \mathcal{M}$,*

$$T_{\varphi(\mathfrak{m})}^{(q)} = T_{\mathfrak{m}}$$

and, for all $v \in T^{(q)}_{\varphi(\mathfrak{m})}$,

$$\mathfrak{b}^{(q)}_v = \varphi(\mathfrak{b}_v).$$

2.5 Probabilistic consequences

Recall the model defined in Equations (1) and (2) for general maps. As promised, we now define its analogue on quadrangulations, and show their equivalence. To that end, we set for all $\mathfrak{m} \in \mathcal{M}_n$, and for all $\mathfrak{q} \in \mathcal{Q}_n$,

$$\mathbb{P}^{\mathrm{map}}_{n,u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}}{[z^n]M(z,u)} \propto u^{b(\mathfrak{m})} \qquad \text{and} \qquad \mathbb{P}^{\mathrm{quad}}_{n,u}(\mathfrak{q}) = \frac{u^{b(\mathfrak{q})}}{[z^n]Q(z,u)} \propto u^{b(\mathfrak{q})},$$

and consider for all $\mathfrak{m} \in \mathcal{M}$ and $\mathfrak{q} \in \mathcal{Q}$ the singular Boltzmann laws (remember that, as explained in Section 2.2, $M(\rho(u),u) = Q(\rho(u),u) < \infty$)

$$\mathbb{P}^{\mathrm{map}}_{u}(\mathfrak{m}) = \frac{u^{b(\mathfrak{m})}\rho(u)^{|\mathfrak{m}|}}{M(\rho(u),u)} \qquad \text{and} \qquad \mathbb{P}^{\mathrm{quad}}_{u}(\mathfrak{q}) = \frac{u^{b(\mathfrak{q})}\rho(u)^{|\mathfrak{q}|}}{Q(\rho(u),u)},$$

then

$$\mathbb{P}^{\mathrm{map}}_{n,u} = \mathbb{P}^{\mathrm{map}}_{u}(\cdot \mid \mathcal{M}_n) \qquad \text{and} \qquad \mathbb{P}^{\mathrm{quad}}_{n,u} = \mathbb{P}^{\mathrm{quad}}_{u}(\cdot \mid \mathcal{Q}_n).$$

By Proposition 5, one has:

Proposition 6. *For all* $\mathfrak{q} \in \mathcal{Q}$ *and* $n \in \mathbb{Z}_{\geqslant 0}$,

$$\mathbb{P}^{\mathrm{quad}}_{n,u}(\mathfrak{q}) = \mathbb{P}^{\mathrm{map}}_{n,u}\left(\varphi^{-1}(\mathfrak{q})\right) \qquad \textit{and} \qquad \mathbb{P}^{\mathrm{quad}}_{u}(\mathfrak{q}) = \mathbb{P}^{\mathrm{map}}_{u}\left(\varphi^{-1}(\mathfrak{q})\right),$$

so, denoting by $*$ *the pushforward, for all* $n \in \mathbb{Z}_{\geqslant 0}$,

$$\mathbb{P}^{\mathrm{quad}}_{n,u} = \varphi_* \mathbb{P}^{\mathrm{map}}_{n,u} \qquad \textit{and} \qquad \mathbb{P}^{\mathrm{quad}}_{u} = \varphi_* \mathbb{P}^{\mathrm{map}}_{u}.$$

2.6 A word on the probabilistic setting

We denote by $\mathbf{M} : \mathcal{M} \to \mathcal{M}$ the canonical random variable on the space of maps, and let $\mathbf{Q} = \varphi(\mathbf{M})$. We denote by $\mathbf{T}$ the block tree associated to $\mathbf{M}$ (and also to $\mathbf{Q}$ by Proposition 5). In this way, under $\mathbb{P}_u$ (resp. $\mathbb{P}_{n,u}$), $\mathbf{M}$ has law $\mathbb{P}^{\mathrm{map}}_u$ (resp. $\mathbb{P}^{\mathrm{map}}_{n,u}$), and, by Proposition 6, $\mathbf{Q}$ has law $\mathbb{P}^{\mathrm{quad}}_u$, (resp. $\mathbb{P}^{\mathrm{quad}}_{n,u}$). Therefore, we will simply use $\mathbb{P}_{n,u}$ and $\mathbb{P}_u$ as a shorthand notation for $\mathbb{P}^{\mathrm{map}}_{n,u}$ and $\mathbb{P}^{\mathrm{map}}_u$.

Maximal simple components of quadrangulations will also be called "blocks" because everything that has been said about blocks (in the sense of maximum 2-connected components) can also be said about the maximum simple quadrangular components of quadrangulations; and likewise in everything that follows. As a consequence, every result about the size of the blocks of a map of size n is valid for blocks of quadrangulations of size n as well.

For v a vertex of $\mathbf{T}$, we denote by $\mathfrak{b}^{\mathbf{M}}_v$ (resp. $\mathfrak{b}^{\mathbf{Q}}_v$) the 2-connected block of $\mathbf{M}$ (resp. simple block of $\mathbf{Q}$) represented by v in $\mathbf{T}$. By Proposition 5, it holds that $\mathfrak{b}^{\mathbf{Q}}_v = \varphi(\mathfrak{b}^{\mathbf{M}}_v)$ for all $v \in \mathbf{T}$, where φ is Tutte's bijection.

These random variables will be studied under probability measures $\mathbb{P}_u$ and $(\mathbb{P}_{n,u})_{n\geqslant 1}$, which were introduced in Section 2.5. We write accordingly $\mathbb{E}_u[\dots]$ and $\mathbb{E}_{n,u}[\dots]$ the expectations with

respect to these probability measures. Unless mentioned otherwise or if it is clear from context, other random variables shall be viewed as defined on some probability space (Ω, P), and the according expectations will be written as $E[\dots]$. In particular we will use the following random variables defined on (Ω, P):

- For each $u \geqslant 0$, the triplet $(\mathbf{T}_{n,u}, \mathbf{M}_{n,u}, \mathbf{Q}_{n,u})$ is $(\mathbf{T}, \mathbf{M}, \mathbf{Q})$ under the law $\mathbb{P}_{n,u}$.
- For each $k \geqslant 1$, the pair $(B_k^{\mathrm{map}}, B_k^{\mathrm{quad}})$ consists of a 2-connected map B_k^{map} with k edges sampled uniformly, together with $B_k^{\mathrm{quad}} = \varphi(B_k^{\mathrm{map}})$ its image by Tutte's bijection. By Proposition 4, the latter is a simple quadrangulation with k faces sampled uniformly.

3 Phase diagram

For μ a probability distribution on $\mathbb{Z}_{\geqslant 0}$ and $n \in \mathbb{Z}_{\geqslant 0}$, we denote by $GW(\mu, n)$ the law of a Galton-Watson tree with offspring distribution μ and conditioned to have n edges. Following [AB19], for $u > 0$ we aim at finding a measure μ^u such that $\mathbf{T}$ under $\mathbb{P}_{n,u}$ has law $GW(\mu^u, 2n)$. To that end, for any $y \in [0, \rho_B]$ we introduce the following probability distribution

$$\mu^{y,u}(2j) := \frac{b_j y^j u^{\mathbb{1}_{j\neq 0}}}{1 + u(B(y) - 1)} \quad \text{and} \quad \mu^{y,u}(2j+1) := 0 \qquad \text{for all } j \in \mathbb{Z}_{\geqslant 0} \tag{9}$$

where b_j and B are defined in Proposition 2. Moreover (see Remark 1 for a discussion), we set

$$y(u) := \rho(u) M^2(\rho(u), u) \quad \text{and} \quad \mu^u := \mu^{y(u),u} \qquad \text{for any } u > 0, \tag{10}$$

where we recall that $\rho(u)$ is the radius of convergence of $z \mapsto M(z, u)$. On Fig. 17, the value of $y(u)$ is represented, using an explicit expression (see Remark 2). Notice that in view of (7), $y(u) \leqslant \rho_B$ for all $u > 0$ and

$$1 + u(B(y(u)) - 1) = M(\rho(u), u). \tag{11}$$

Then, by (5), for all $u > 0$, we have:

$$\mu^u(\{2j\}) \sim \sqrt{\frac{3}{\pi}} \frac{2}{27} \frac{u}{M(\rho(u), u)} \left(\frac{27}{4} y(u)\right)^j j^{-5/2}, \quad \text{as } j \to \infty,$$

so that by setting

$$c(u) = \sqrt{\frac{3}{\pi}} \frac{2}{27} \frac{u}{M(\rho(u), u)}, \tag{12}$$

it holds that

$$\mu^u(\{2j\}) \sim c(u) \left(\frac{27}{4} y(u)\right)^j j^{-5/2}, \quad \text{as } j \to \infty. \tag{13}$$

The following proposition extends [AB19, Proposition 3.1] to our setting.

Proposition 7. *Let $(\mathbf{B}_v, v \in \mathbf{T})$ be either the family $(\mathfrak{b}_v^{\mathbf{M}})_{v\in\mathbf{T}}$ of blocks of $\mathbf{M}$, or $(\mathfrak{b}_v^{\mathbf{Q}})_{v\in\mathbf{T}}$ of blocks of $\mathbf{Q}$. For every $u > 0$, under $\mathbb{P}_u$, the law of tree of blocks $(\mathbf{T}, (\mathbf{B}_v, v \in \mathbf{T}))$ can be described as follows.*

- *$\mathbf{T}$ follows the law $GW(\mu^u)$;*

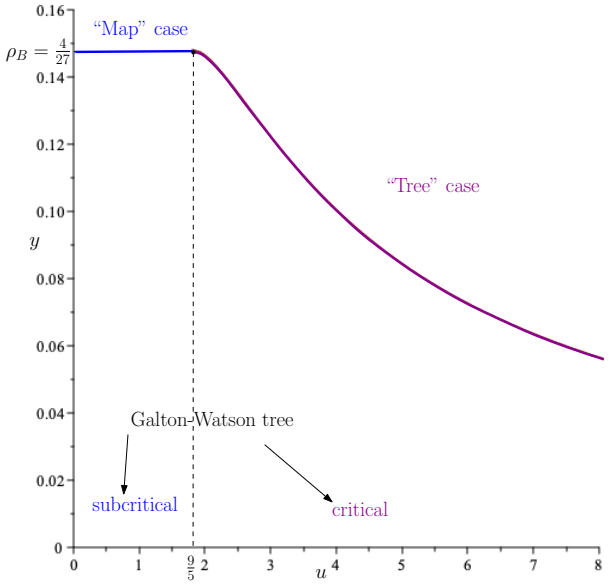

Figure 17: Plot of y as a function of u.

- *Conditionally given $\mathbf{T} = \mathfrak{t}$, the blocks $(\mathbf{B}_v, v \in \mathfrak{t})$ are independent random variables, and, for $v \in \mathfrak{t}$, $\mathbf{B}_v$ follows a uniform distribution on the set of blocks of size $k_v(\mathfrak{t})/2$, where $k_v(\mathfrak{t})$ is the number of children of v in $\mathfrak{t}$.*

For every $n \geqslant 1$, the same statements hold under $\mathbb{P}_{n,u}$, only replacing $GW(\mu^u)$ with $GW(\mu^u, 2n)$.

Proof. It suffices to prove the statement for $\mathbb{P}_u$ as, by Proposition 3, the block-tree of a map of size n has size $2n$.

Let $\mathfrak{t}$ be a tree where each vertex has an even number of children, and let $(\mathfrak{b}_v, v \in \mathfrak{t})$ be a family of (2-connected, or simple) blocks, with $2|\mathfrak{b}_v| = k_v(\mathfrak{t})$ for any $v \in \mathfrak{t}$. Let $\mathfrak{m}$ be the map (or quadrangulation) with block decomposition given by $(\mathfrak{t}, (\mathfrak{b}_v, v \in \mathfrak{t}))$.

Then, we have

$$\begin{aligned}
&\mathbb{P}_u\left(\mathbf{T} = \mathfrak{t}, \mathbf{B}_v = \mathfrak{b}_v \ \forall v \in \mathfrak{t}\right) = \mathbb{P}_u(\mathfrak{m}) \\
&= \frac{\rho(u)^{|\mathfrak{m}|} u^{b(\mathfrak{m})}}{M(\rho(u), u)} = \frac{\rho(u)^{\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})/2} u^{\sum_{v\in\mathfrak{t}} \mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M(\rho(u), u)} \prod_{v\in\mathfrak{t}} \frac{b_{\frac{k_v(\mathfrak{t})}{2}}}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \frac{1}{M(\rho(u), u)} \left(\frac{y(u)}{M^2(\rho(u), u)}\right)^{\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})/2} \prod_{v\in\mathfrak{t}} b_{\frac{k_v(\mathfrak{t})}{2}} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \frac{\prod_{v\in\mathfrak{t}} b_{\frac{k_v(\mathfrak{t})}{2}} y(u)^{k_v(\mathfrak{t})/2} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M(\rho(u), u)^{1+\sum_{v\in\mathfrak{t}} k_v(\mathfrak{t})}} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}} \\
&= \prod_{v\in\mathfrak{t}} \frac{b_{\frac{k_v(\mathfrak{t})}{2}} y(u)^{k_v(\mathfrak{t})/2} u^{\mathbb{1}_{k_v(\mathfrak{t})\neq 0}}}{M(\rho(u), u)} \times \prod_{v\in\mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}}
\end{aligned}$$

$$= GW(\mu^u)(\mathfrak{t}) \times \prod_{v \in \mathfrak{t}} \frac{1}{b_{\frac{k_v(\mathfrak{t})}{2}}}.$$

This concludes the proof. □

Theorem 1. *Recall the definition of $c(u)$ given in (12). Then, depending on the value of u, the model $\mathbb{P}_u$ undergoes the following phase transition, driven by the properties of μ^u:*

Subcritical case. *For $u < u_C := 9/5$,*

$$E(u) := E[\mu^u] = \frac{8u}{3(3+u)} < 1 \qquad \text{and} \qquad \mu^u(\{2j\}) \sim c(u) j^{-5/2} \tag{14}$$

where $c(u) = \sqrt{\frac{3}{\pi}} \frac{2u}{9(3+u)}$;

Critical case. *For $u = u_C := 9/5$,*

$$E[\mu^{u_C}] = 1 \qquad \text{and} \qquad \mu^{u_C}(\{2j\}) \sim \frac{1}{4\sqrt{3\pi}} j^{-5/2};$$

Supercritical case. *For $u > 9/5$,*

$$E[\mu^u] = 1 \qquad \text{and} \qquad \mu^u(\{2j\}) \sim c(u) \left(\frac{27}{4} y(u)\right)^j j^{-5/2},$$

where $y(u) < 4/27$ so that μ^u has exponential moments.

Notice that the case $u = 1$, which corresponds to uniform planar maps, as studied by Addario-Berry [AB19], falls in the subcritical regime.

Proof of Theorem 1. Let us first explain how the value $u_C := 9/5$ appears. Let $u > 0$ and $y \in (0, 4/27]$. By (9),

$$E[\mu^{y,u}] = \sum_{j \in \mathbb{Z}_{\geqslant 0}} \frac{2j b_j y^j u^{\mathbb{1}_{j \neq 0}}}{1 + u(B(y) - 1)} = \frac{2uyB'(y)}{1 + u(B(y) - 1)}. \tag{15}$$

It follows that

$$E[\mu^{y,u}] = 1 \Leftrightarrow u = \frac{1}{2yB'(y) - B(y) + 1}. \tag{16}$$

The mapping $y \in (0, 4/27] \mapsto d(y) := 2yB'(y) - B(y) + 1$ is increasing. Indeed, for all $y \in (0, 4/27]$,

$$d(y) = \sum_{n \geqslant 1} 2n b_n y^n - \sum_{n \geqslant 0} b_n y^n + 1 = \sum_{n \geqslant 1} (2n-1) b_n y^n.$$

Moreover, if follows from (6) that $d(0) = 0$ and $d(4/27) = 5/9$. So $1/d(y)$ maps bijectively $(0, 4/27]$ to $[9/5, +\infty)$. Therefore, there exists $y \in (0, 4/27]$ such that the law $\mu^{y,u}$ is critical if and only if $u \in [9/5, +\infty)$, and this y is unique.

We now conclude the proof of the theorem. For the sake of completeness, we recall an argument from [Bon16, §8.2.2]. Recall (7):

$$M(z,u) = uB(zM(z,u)^2) + 1 - u.$$

For a fixed u, there are two possible sources of singularity:

1. The pair $(z_0 = \rho(u), m_0 = M(\rho(u), u))$ satisfies $\frac{\partial H}{\partial m}(z_0, m_0) = 0$ for $H : (z,m) \mapsto m - uB(zm^2) - 1 + u$, thus being a singularity by the contraposition of the implicit function theorem. In this case,

$$1 - 2\rho(u)M(\rho(u),u)uB'(\rho(u)M^2(\rho(u),u)) = 0, \quad \text{so} \quad 2\rho(u)M(\rho(u),u)uB'(y(u)) = 1.$$

   Then, by (11),

$$2y(u)B'(y(u)) - B(y(u)) + 1 = \frac{2\rho(u)M^2(\rho(u),u)}{2u\rho(u)M(\rho(u),u)} - \frac{M(\rho(u),u) + u - 1}{u} + 1 = \frac{1}{u}, \tag{17}$$

   which is to say that $y(u) = \rho(u)M^2(\rho(u),u)$ satifies (16). This is possible if and only if $u \geqslant 9/5$. Then, it follows that $E[\mu^u] = 1$, and (13) gives the asymptotic behaviour of $\mu^u(2j)$.

2. A singularity of B is reached so $\rho(u)M^2(\rho(u),u) = \rho_B = 4/27$ *i.e.* $y(u) = 4/27$. Then, the value of $E(u)$ is obtained as an immediate consequence of Equations (6) and (15), and the asymptotic behaviour of $\mu^u(2j)$ comes from Equations (12) and (13). This happens iff $u \leqslant 9/5$.

Notice that at $u = u_C$, both types of singularity are reached. □

Remark 1. *The proof of Theorem 1 highlights the reasons behind our choice of $y(u)$ in (10). When $u \geqslant 9/5$, we choose $y(u)$ such that $E(u) = 1$. When $u < 9/5$, this is not possible, and we choose the value of $y(u)$ maximising $E(u)$ so that, when conditioning the trees to be of size $2n$, the conditioning is as little degenerated as possible. See [Jan12, §7] for further details.*

Remark 2. *Using (16), we obtain an explicit expression for y in terms of u for $u \geqslant u_C$. By [Tut63], the series B is algebraic and for all $y \in [0, 4/27]$,*

$$B(y)^3 - B(y)^2 - 18yB(y) + 27y^2 + 16y = 0. \tag{18}$$

This gives an expression of B' in terms of B, and taking the resultant between this new equation and (16) allows to eliminate B. Initial conditions then give

$$u = \frac{1}{2yB'(y) - B(y) + 1} \Leftrightarrow y = \left(1 - \sqrt{1 - \frac{1}{u}}\right)\left(1 - \frac{1}{u}\right). \tag{19}$$

4 Study of the size of the largest blocks

4.1 Subcritical case

To investigate the distribution of the size of the largest blocks, in the subcritical case, we follow the approach developped in [AB19], which consists in studying the degrees in the block tree of a map. To that end, we rely on results of *condensation* in Galton-Watson trees: exactly one of the nodes

has a degree linear in the size. To that end, we rely on Janson's survey [Jan12], in which there is a refinement of the study of the largest degree of a subcritical Galton-Watson tree with condensation by Jonsson and Stefánsson [JS11]. The condensation phenomenon is visible in the following result where, denoting by d_{TV} the total variation distance, we write $X_n \overset{(d)}{\approx} Y_n$ if $d_{TV}(X_n, Y_n) \to 0$ as $n \to \infty$:

Proposition 8 ([Jan12, Theorem 19.34]). *Let μ be a probability distribution on $\mathbb{Z}_{\geqslant 0}$ such that $\mu(0) > 0$, $E[\mu] < 1$ and there exists c satisfying $\mu(k) \sim_{k\to\infty} ck^{-5/2}$. Let $D_{n,1} \geqslant D_{n,2} \geqslant \ldots \geqslant D_{n,n}$ be the ranked list of the number of children of a μ-Galton-Watson tree conditioned to have n edges. Then, letting $\xi_1, \ldots, \xi_{n-1}$ be a family of $n-1$ independent random variables of law μ and $\left(\xi_1^{(n)}, \ldots, \xi_{n-1}^{(n)}\right)$ their decreasing reordering, it holds that:*

$$(D_{n,1}, \ldots, D_{n,n}) \overset{(d)}{\approx} \left(n - \sum_{i=1}^{n-1} \xi_i, \xi_1^{(n)}, \ldots, \xi_{n-1}^{(n)}\right). \tag{20}$$

We combine this proposition with the fact that $\mathbf{T}$ is a Galton-Watson tree under $\mathbb{P}_u$ to get the following generalization of [AB19, Theorem 3.3][4] to every value of $u \in (0, 9/5)$. This is a rephrasing of the results for trees of [Jan12], to which we add the proof of the joint convergence. For $\mathfrak{m}$ a map of size n, denote by $\mathrm{LB}_1(\mathfrak{m}) \geqslant \ldots \geqslant \mathrm{LB}_{b(\mathfrak{m})}(\mathfrak{m})$ the sizes of its blocks in decreasing order. By convention, we set $\mathrm{LB}_k(\mathfrak{m}) = 0$ if $k > b(\mathfrak{m})$.

Theorem 2. *Let $u \in (0, 9/5)$. Recall that $E(u)$ and $c(u)$ are defined in Equations (12) and (14). Then,*

$$\mathrm{LB}_1(\mathbf{M}_{n,u}) = (1 - E(u))n + O_{\mathbb{P}}(n^{2/3}) \quad \text{and} \quad \mathrm{LB}_2(\mathbf{M}_{n,u}) = O_{\mathbb{P}}(n^{2/3}).$$

Moreover, the following joint convergence holds:

$$\left(\frac{1}{2nc(u)}\right)^{2/3} ((1 - E(u))n - \mathrm{LB}_1(\mathbf{M}_{n,u}), (\mathrm{LB}_j(\mathbf{M}_{n,u}), j \geqslant 2)) \xrightarrow[n\to\infty]{(d)} \left(L_1, \left(\Delta L_{(j-1)}, j \geqslant 2\right)\right) \tag{21}$$

where $(L_t)_{t\in[0,1]}$ is a Stable process of parameter $3/2$ such that

$$E\left[e^{-sL_1}\right] = e^{\Gamma(-3/2)s^{3/2}}$$

and $\Delta L_{(1)} \geqslant \Delta L_{(2)} \geqslant \ldots$ is the ranked sequence of its jumps.

When $u \to 0$, we have $1 - E(u) \to 1$: as expected, if the map has only one block, its size is n.

Remark 3. *If $(L_t)_{t\in[0,1]}$ is a Stable process of parameter $3/2$ satisfying $E\left[e^{-sL_1}\right] = e^{\Gamma(-3/2)s^{3/2}}$ for s such that $Re(s) \geqslant 0$; then, it is known that (see [Ber96, Theorem 1] and its proof):*

$$L_1 \overset{(d)}{=} \lim_{\varepsilon\to 0} \sum_{j:\Delta L_{(j)} \geqslant \varepsilon} \Delta L_{(j)} - \frac{2}{\sqrt{\varepsilon}}.$$

[4] One may notice that our $c(1)$ differs from Addario-Berry's c which is because there is a small miscalculation for c in [AB19] due to the fact that $T_{n,u}$ does not have n edges but $2n$ edges.

Proof of Theorem 2. Recall that the subcritical case corresponds to $u \in (0, 9/5)$, for which we have

$$\rho(u)M^2(\rho(u), u) = 4/27.$$

We follow essentially the same lines of proof as in [AB19], but refining the arguments so as to establish the joint convergence stated in (21). Theorem 1 shows that the hypotheses of Proposition 8 are satisfied in the subcritical case.

Let $(\xi_i)_{i\geqslant 1}$ be a family of iid random variables of law μ^u and let $\left(\xi_1^{(n)}, \dots, \xi_n^{(n)}\right)$ be the decreasing reordering of its first n variables (take the convention $\xi_i^{(n)} = 0$ if $i > n$). Let us consider the following cumulative process:

$$L_t^{(n)} = \frac{\sum_{i=1}^{\lceil 2nt \rceil} \xi_i - 2ntE(u)}{C(u)(2n)^{2/3}} \quad \text{for } t \in [0,1], \quad \text{where} \quad C(u) = 2c(u)^{2/3}.$$

It is standard [Fel71, Theorem XVII.5.2] [JS87, Chapter VII, Corollary 3.6] that there exists a Lévy process $(L_t)_{t\in[0,1]}$ with Lévy measure $\pi(dx) = x^{-5/2}dx\mathbb{1}_{\{x>0\}}$ so that for s such that $Re(s) \geqslant 0$,

$$E\left[e^{-sL_1}\right] = e^{\Gamma(-3/2)s^{3/2}},$$

and such that the following convergence holds in the Skorokhod topology

$$\left(L_t^{(n)}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (L_t)_{t\in[0,1]}\,. \tag{22}$$

By definition of the process $L_t^{(n)}$, $\frac{\xi_i}{C(u)(2n)^{2/3}}$ is its i-th jump. In particular, denoting by ΔP_t the jump of the process (P_t) at time t (which may equal 0),

$$\frac{\xi_1^{(2n)}}{C(u)(2n)^{2/3}} = \sup_{0\leqslant t\leqslant 1} \Delta L_t^{(n)}.$$

By [JS87, Chapter VI, Proposition 2.4], (22) gives

$$\frac{\xi_1^{(2n)}}{C(u)(2n)^{2/3}} \xrightarrow[n\to\infty]{(d)} \sup_{0\leqslant t\leqslant 1} \Delta L_t := \Delta L_{(1)}.$$

By construction of a Lévy process, $(\Delta L_{(j)})_{j\geqslant 1}$ has same law as the decreasing rearrangement of the atoms of a Poisson random measure with intensity π on $\mathbb{R}^+$ (see e.g. [Ber96, Theorem 1]). By denoting $t_1^{(n)}$ the time at which the jump $\xi_1^{(2n)}$ of the process $L_t^{(n)}$ is realised, one has:

$$\frac{\xi_2^{(2n)}}{C(u)(2n)^{2/3}} = \sup_{0\leqslant t\leqslant 1} \Delta\left(L_t^{(n)} - \frac{\xi_1^{(2n)}}{C(u)(2n)^{2/3}}\mathbb{1}_{t\geqslant t_1^{(n)}}\right)_t.$$

So, applying again [JS87, Chapter VI, Proposition 2.4], one gets, denoting by t_1 the time of the largest jump of (L_1):

$$\frac{\xi_2^{(2n)}}{C(u)(2n)^{2/3}} \xrightarrow[n\to\infty]{(d)} \sup_{0\leqslant t\leqslant 1} \Delta\left(L_t - \Delta L_{(1)}\mathbb{1}_{t\geqslant t_1}\right)_t = \Delta L_{(2)}.$$

It is again possible to iterate by subtracting the largest jump: for all $k \geqslant 1$,

$$\frac{1}{C(u)(2n)^{2/3}} \left(\xi_1^{(2n)}, \dots, \xi_k^{(2n)}\right) \xrightarrow[n\to\infty]{(d)} \left(\Delta L_{(1)}, \dots, \Delta L_{(k)}\right). \tag{23}$$

However, by Proposition 8 and (20), one has (recall that a map of size n has $2n+1$ components, some of which might be empty):

$$2\left(\mathrm{LB}_1(\mathbf{M}_{n,u}), \dots, \mathrm{LB}_{2n+1}(\mathbf{M}_{n,u})\right) \overset{(d)}{\approx} \left(2n - \sum_{i=1}^{2n} \xi_i, \xi_1^{(2n)}, \dots, \xi_{2n}^{(2n)}\right).$$

Therefore, for all $k \geqslant 2$ fixed

$$\begin{aligned}
&\left(\frac{(1-E(u))n - \mathrm{LB}_1(\mathbf{M}_{n,u})}{\frac{1}{2}C(u)(2n)^{2/3}}, \frac{\mathrm{LB}_2(\mathbf{M}_{n,u})}{\frac{1}{2}C(u)(2n)^{2/3}}, \dots, \frac{\mathrm{LB}_k(\mathbf{M}_{n,u})}{\frac{1}{2}C(u)(2n)^{2/3}}\right) \\
&\overset{(d)}{\approx} \left(\frac{\sum_{i=1}^{2n}\xi_i - 2E(u)n}{C(u)(2n)^{2/3}}, \frac{\xi_1^{(2n)}}{C(u)(2n)^{2/3}}, \dots, \frac{\xi_k^{(2n)}}{C(u)(2n)^{2/3}}\right) \\
&\xrightarrow[n\to\infty]{(d)} \left(L_1, \Delta L_{(1)}, \dots, \Delta L_{(k)}\right).
\end{aligned}$$

This allows to conclude since k is arbitrary.

□

4.2 Supercritical case

The supercritical case corresponds to $u \in (9/5, +\infty)$ and $y(u) = \rho(u)M^2(\rho(u), u) \in (0, 4/27)$. Recall that in this case T is distributed under $\mathbb{P}_u$ as a critical Galton-Watson tree with finite exponential moments by Proposition 7 and Theorem 1.

Properties of the maximum degree of critical Galton-Watson trees have been extensively studied by Janson [Jan12], building on work by Meir and Moon [MM90]. For the case where the offspring distribution admits finite exponential moments, Janson shows the following result.

Proposition 9 ([Jan12, Theorem 19.16]). *Let μ be a probability distribution on $\mathbb{Z}_{\geqslant 0}$ such that $\mu(0) > 0$, and $\mu(k+1)/\mu(k)$ converges to a finite limit as $k \to \infty$. Let $D_{n,i}$ be the i-th maximal number of children of nodes in a μ-Galton-Watson tree conditioned to have n edges. Denote by ρ the radius of convergence of $\Phi : t \mapsto \sum_{k\in\mathbb{Z}_{\geqslant 0}} \mu(k)t^k$, and $\nu = \lim_{x\to\rho^-} x\frac{\Phi'(x)}{\Phi(x)}$. Suppose $\nu > 1$. Then, denoting $k(n) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid \mu(k) \geqslant 1/n\}$, for all $j \geqslant 1$,*

$$D_{n,j} = k(n) + O_{\mathbb{P}}(1).$$

In our case, the asymptotic of $k(n)$ can be computed thanks to results about the Lambert W function, which is the compositional inverse of $x \in \mathbb{R} \mapsto xe^x \in [-e^{-1}, +\infty)$. This gives the following theorem.

Theorem 3. *Let $u > u_C$. For all fixed $j \geqslant 1$, it holds as $n \to \infty$ that*

$$\mathrm{LB}_j(\mathbf{M}_{n,u}) = \frac{\ln(n)}{2\ln\left(\frac{4}{27y(u)}\right)} - \frac{5\ln(\ln(n))}{4\ln\left(\frac{4}{27y(u)}\right)} + O_{\mathbb{P}}(1).$$

Proof. The probability $\mu^u(\{2k\})$ is decreasing with k. So, by (13), for n large enough, to study $k(n)$ it is sufficient to study for which k one has

$$c(u)\left(\frac{27}{4}\rho(u)M^2(\rho(u),u)\right)^k k^{-5/2}\left(1+o(1)\right) \geqslant \frac{1}{n}.$$

For sake of compactness, set $w(u) = \left(\frac{27}{4}\rho(u)M^2(\rho(u),u)\right)^{-1} = \left(\frac{27}{4}y(u)\right)^{-1}$. Note that $w(u) > 1$ since $u > u_C$. Consequently, the previous inequality is equivalent to

$$w(u)^k k^{5/2} \leqslant c(u)n\left(1+o(1)\right).$$

Notice that this is equivalent to

$$\frac{2}{5}\ln(w(u))k \cdot e^{\frac{2}{5}\ln(w(u))k} \leqslant \frac{2}{5}\ln(w(u))(nc(u))^{2/5}\left(1+o(1)\right).$$

Therefore, $k(n)$ is the largest integer such that:

$$\frac{2}{5}\ln(w(u))k(n) \leqslant W\left(\frac{2}{5}\ln(w(u))(nc(u))^{2/5}\left(1+o(1)\right)\right)$$

where W denotes the Lambert W function. It is known that W satisfies, for $x \to \infty$,

$$W(x) = \ln(x) - \ln(\ln(x)) + o(1),$$

which concludes the proof. □

4.3 Critical case

The critical case corresponds to $u = 9/5$ and $\rho(u)M^2(\rho(u),u) = 4/27$. As shown in Theorem 1, the offspring distribution has a power law tail in $cj^{-\alpha-1}$, where $\alpha = 3/2 \in (1,2)$. In this case, the variance is infinite, so that the method of Section 4.2 cannot be used. However, this case is directly treated in Janson's survey [Jan12, Example 19.27 and Remark 19.28].

Theorem 4. *The following convergence holds:*

$$\left(\frac{\mathrm{LB}_j(\mathbf{M}_{n,u_c})}{n^{2/3}}, j \geqslant 1\right) \xrightarrow[n\to\infty]{(d)} \left(E_{(j)}, j \geqslant 1\right),$$

where the $\left(E_{(j)}\right)$ are the ordered atoms of a Point Process E on $[0,\infty]$, satisfying that the random variable $E_{a,b} = \#\left(E \cap [a,b]\right)$ has a probability generating function convergent for all $z \in \mathbb{C}$ with

$$E\left[z^{E_{a,b}}\right] = \frac{1}{2\pi g(0)}\int_{-\infty}^{\infty} \exp\left(c\Gamma(-3/2)(-it)^{3/2} + (z-1)c\int_a^b x^{-5/2}e^{itx}dx\right)dt,$$

where

$$g : x \mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt + c\Gamma(-3/2)(-it)^{3/2}} dt.$$

The intensity measure π of E satisfies, for $x > 0$,

$$\pi(dx) = cx^{-5/2} \frac{g(-x)}{g(0)} dx,$$

and, for all $j \geqslant 1$,

$$E_{(j)} > 0 \qquad \text{almost surely.}$$

Remark 4. *By Theorem 1 and [Duq03, Proposition 4.3][5], one has the convergence of the (appropriately) rescaled Łukasiewicz path of $\mathbf{T}_{n,u_C}$ towards a 3/2-stable excursion. Therefore, using [JS87, Chapter VI, Proposition 2.4] and following the same line of arguments as in the proof of Theorem 2, one gets that the $E_{(j)}$ are distributed like the reordered jumps of a 3/2-stable excursion (multiplied by a constant factor).*

5 Scaling limits

The preceding sections exhibited, *via* a study of the block-tree, a phase transition of a combinatorial nature, in terms of the size of the largest blocks, when the parameter u reaches $u_C = 9/5$, both for the model on general maps and the one on quadrangulations. The goal of the present section is to expand on this phase transition by considering metric properties of the models in each phase, in the sense of taking *scaling limits*, see Section 5.1 for definitions.

Because Tutte's bijection commutes with the block decomposition of both models under consideration, as stated in Proposition 5, the combinatorial picture of Section 4 is the same for both models. However, obtaining global metric properties under either model requires a good understanding of the metric behaviour of the underlying blocks. As of now, the required results exist only for simple quadrangulations. Consequently, our scaling limit results are complete only for the quadrangulation model.

In Section 5.1, we introduce the relevant formalism to state our scaling limit results, as well as a deviation estimate for the diameters of blocks, which will be useful for all values of u.

In Section 5.2, we prove Theorem 5, which identifies scaling limits simultaneously when $u > u_C$ and $u = u_C$. For both models, there is convergence after suitable rescaling to a random continuous tree, namely a *Brownian tree* when $u > u_C$ and a 3/2 *stable tree* when $u = u_C$. This convergence holds in the Gromov-Hausdorff-Prokhorov (GHP) sense – between measured metric spaces – when maps and quadrangulations are equipped with the uniform measure on their vertices.

Finally in Section 5.3, we prove Theorem 6 which deals with the GHP scaling limit when $u < u_C$. In this phase, the one-big-block identified in Theorem 2 converges after rescaling to a scalar multiple of the *Brownian sphere*, and the contribution of all other blocks is negligible. This result is proved only for the quadrangulation model since it relies crucially on the scaling limit result for uniform simple quadrangulations obtained in [ABA17]. No such result is available yet for uniform 2-connected general maps, although one expects that it should hold.

[5] The result is stated under an aperiodicity hypothesis for the reproduction law, which can be omitted; see the discussion in the proof of Proposition 14.

5.1 Preliminaries

5.1.1 The Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov topologies

Originating from the ideas of Gromov, the following notions of metric geometry have become widely used in probability theory to state scaling limit results. We refer the interested reader to [BBI01] for general background on metric geometry and [Mie07, Section 6] for an exposition of the main properties of the Gromov-Hausdorff and Gromov-Hausdorff-Prokhorov topologies, and especially their definition *via* correspondences and couplings that we use here.

Define a *correspondence* between two sets X and Y as a subset C of $X \times Y$ such that for all $x \in X$, there exists $y \in Y$ such that $(x, y) \in C$, and *vice versa*. The set of correspondences between X and Y is denoted as $\mathrm{Corr}(X, Y)$. If (X, d_X) and (Y, d_Y) are compact metric spaces and $C \in \mathrm{Corr}(X, Y)$ is a correspondence, one may define its *distortion*:

$$\mathrm{dis}(C; d_X, d_Y) = \sup\Big\{|d_X(x, \widetilde{x}) - d_Y(y, \widetilde{y})| \colon (x, y) \in C,\ (\widetilde{x}, \widetilde{y}) \in C\Big\}.$$

This allows to define the *Gromov-Hausdorff* distance between (isometry classes of) compact metric spaces

$$d_{\mathrm{GH}}\big((X, d_X), (Y, d_Y)\big) = \frac{1}{2} \inf\Big\{\mathrm{dis}(C; d_X, d_Y) \colon C \in \mathrm{Corr}(X, Y)\Big\}.$$

One can modify this notion of distance in order to get a distance between compact measured metric spaces. For measured spaces (X, ν_X) and (Y, ν_Y) such that ν_X and ν_Y are *probability* measures, let us denote by $\mathrm{Coupl}(\nu_X, \nu_Y)$ the set of couplings between ν_X and ν_Y, *i.e.* the set of measures γ on $X \times Y$ with respective marginals ν_X and ν_Y. Then the *Gromov-Hausdorff-Prokhorov* distance is defined as

$$\begin{aligned} &d_{\mathrm{GHP}}\big((X, d_X, \nu_X), (Y, d_Y, \nu_Y)\big) \\ &\qquad = \inf\Big\{\max\Big(\tfrac{1}{2}\,\mathrm{dis}(C; d_X, d_Y), \gamma\big((X \times Y) \setminus C\big)\Big) \colon C \in \mathrm{Corr}(X, Y),\ \gamma \in \mathrm{Coupl}(\nu_X, \nu_Y)\Big\}. \end{aligned}$$

When (X, d_X) and (Y, d_Y) are the same metric space, one can bound this distance by the *Prokhorov distance* between the measures ν_X and ν_Y. This distance is defined for ν_1 and ν_2 two Borel measures on the same metric space (X, d) by

$$d_{\mathrm{P}}^{(X,d)}(\nu_1, \nu_2) = \inf\Big\{\varepsilon > 0 \colon \nu_1(A) \leqslant \nu_2(A^\varepsilon) + \varepsilon \text{ and } \nu_2(A) \leqslant \nu_1(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{B}(X)\Big\}, \tag{24}$$

where A^ε is the set of points $x \in X$ such that $d(x, A) < \varepsilon$. The bound mentioned above then corresponds to the inequality

$$d_{\mathrm{GHP}}\big((X, d, \nu_1), (X, d, \nu_2)\big) \leqslant d_{\mathrm{P}}^{(X,d)}(\nu_1, \nu_2), \tag{25}$$

which is a consequence of Strassen's Theorem, see [Dud02, Section 11.6].

Finally we will use the following fact, the proof of which is left to the reader. For $a \in [0, 1)$ and Borel probability measures μ, ν and ν' on some metric space (X, d), it holds that

$$d_{\mathrm{P}}^{(X,d)}\big(a\mu + (1-a)\nu, a\mu + (1-a)\nu'\big) = (1-a) d_{\mathrm{P}}^{(X,d)}(\nu, \nu'). \tag{26}$$

5.1.2 Formulation of the GHP-scaling limit problem

Let us begin by setting the notation for the measured metric spaces that one can canonically associate to the combinatorial objects under consideration. We associate to a tree (resp. map or quadrangulation) the following measured metric spaces:

- For $\mathfrak{t}$ a *tree*, denote by $V(\mathfrak{t})$ the set of its vertices, by $d_\mathfrak{t}$ the distance that the graph distance induces on $V(\mathfrak{t})$, by $\nu_\mathfrak{t}$ the uniform probability measure on $V(\mathfrak{t})$ and by $\underline{\mathfrak{t}}$ the measured metric space $\underline{\mathfrak{t}} = (V(\mathfrak{t}), d_\mathfrak{t}, \nu_t)$. Recall that for $v \in \mathfrak{t}$, the number of children of v is denoted by $k_v(\mathfrak{t})$.
- For $\mathfrak{m}$ a *map*, recall that $V(\mathfrak{m})$ is its vertex set, and denote by $d_\mathfrak{m}$ the graph distance on $V(\mathfrak{m})$, by $\nu_\mathfrak{m}$ the uniform probability measure on $V(\mathfrak{m})$ and by $\underline{\mathfrak{m}}$ the measured metric space $\underline{\mathfrak{m}} = (V(\mathfrak{m}), d_\mathfrak{m}, \nu_m)$.
- For $\mathfrak{q}$ a *quadrangulation*, denote by $V(\mathfrak{q})$ its vertex set, by $d_\mathfrak{q}$ the graph distance on $V(\mathfrak{q})$, by $\nu_\mathfrak{q}$ the uniform probability measure on $V(\mathfrak{q})$ and by $\underline{\mathfrak{q}}$ the measured metric space $\underline{\mathfrak{q}} = (V(\mathfrak{q}), d_\mathfrak{q}, \nu_q)$.

The problem of finding a GHP-scaling limit consists in finding a suitable rescaling of a sequence of random compact measured metric spaces so that it admits a non-trivial limit in distribution for the GHP-topology. Let us introduce a convenient notation for the rescaling operation on a measured metric space. For $\underline{X} = (X, d, \nu)$ a measured metric space and $\lambda > 0$, we denote by $\lambda \cdot \underline{X}$ the measured metric space $(X, \lambda d, \nu)$.

5.1.3 A useful deviation estimate

We shall now prove a deviation estimate for the diameters of the blocks of **M** and **Q**. It will prove useful for all values of $u > 0$. We recall the definition of *stretched-exponential* quantities, as this notion provides a concise way to deal with the probabilities of exceptional events.

Definition 8. *A sequence (p_n) of real numbers is said to be* stretched-exponential *as $n \to \infty$ if there exist constants $\gamma, C, c > 0$ such that*

$$|p_n| \leqslant C \exp(-cn^\gamma).$$

As is evident from the definition, if $(p_n)_n$ and $(q_n)_n$ are stretched-exponential sequences, then so are the sequences $(p_n + q_n)_n$, $(p_n q_n)_n$, $(n^\alpha p_n)_n$ and $(n^\alpha \sup_{k \geqslant n^\beta} p_k)_n$ with arbitrary $\alpha, \beta > 0$.

The input we shall rely on to derive our estimate is a deviation estimate for the diameter of **one** block, in both the case of 2-connected blocks of maps and simple blocks of quadrangulations.

Proposition 10. *For any $\varepsilon > 0$, the probabilities*

$$P\left(\mathrm{diam}(B_k^{\mathrm{map}}) \geqslant k^{1/4+\varepsilon}\right) \qquad \text{and} \qquad P\left(\mathrm{diam}(B_k^{\mathrm{quad}}) \geqslant k^{1/4+\varepsilon}\right)$$

are stretched-exponential as $k \to \infty$.

Proof. The estimate for uniform 2-connected maps $(B_k^{\mathrm{map}})_{k \geqslant 0}$ is obtained from [CFGN15, Theorem 3.7, specialized to $x = 1$]. To obtain the estimate for uniform simple blocks of quadrangulations $(B_k^{\mathrm{quad}})_{k \geqslant 0}$, one easily checks that for any path of length $l \geqslant 0$ in a map $\mathfrak{m}$, there exists a path

with same endpoints and length at most $2l$ in $\varphi(\mathfrak{m})$, its image by Tutte's bijection. Therefore for every map $\mathfrak{m}$ one has $\mathrm{diam}(\varphi(\mathfrak{m})) \leqslant 2\,\mathrm{diam}(\mathfrak{m})$. In particular $\mathrm{diam}(B_k^{\mathrm{quad}}) \leqslant 2\,\mathrm{diam}(B_k^{\mathrm{map}})$, and the conclusion follows from the estimate for $(B_k^{\mathrm{map}})_{k\geqslant 0}$ □

This deviation estimate for the diameter of **one** block allows to control the deviations of the diameter of **every** block of $\mathbf{M}_{n,u}$ and $\mathbf{Q}_{n,u}$, in the sense of the following corollary.

Corollary 11. *For all $u > 0$ and all $\delta > 0$, the probabilities*

$$P\left(\exists v \in \mathbf{T}_{n,u},\, \mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \geqslant \max\left(n^{1/6}, k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}\right)\right), \quad n \geqslant 1, \tag{27}$$

$$P\left(\exists v \in \mathbf{T}_{n,u},\, \mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \geqslant \max\left(n^{1/6}, k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}\right)\right), \quad n \geqslant 1, \tag{28}$$

are stretched-exponential as $n \to \infty$.

Proof. Let $\mathfrak{b}$ be either a 2-connected map, or a simple quadrangulation. Then $\mathrm{diam}(\mathfrak{b})$ is bounded by its number of edges, which is $|\mathfrak{b}|$ if $\mathfrak{b}$ is a map, and $2|\mathfrak{b}|$ if it is a quadrangulation. In particular, recalling that the outdegrees in the block-tree are twice the sizes of the respective blocks, we get for all $u > 0$ and $n \geqslant 1$, that

$$\forall v \in \mathbf{T}_{n,u},\ \left[\mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \leqslant k_v(\mathbf{T}_{n,u})/2\right] \text{ and } \left[\mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \leqslant 2 \cdot k_v(\mathbf{T}_{n,u})/2\right].$$

Denote by $\mathrm{A}(\mathbf{M}_{n,u})$ the "bad" subset of $\mathbf{T}_{n,u}$ made of the vertices v such that both $k_v(\mathbf{T}_{n,u})/2 \geqslant n^{1/6}$ and $\mathrm{diam}(\mathfrak{b}_v^{\mathbf{M}_{n,u}}) \geqslant k_v(\mathbf{T}_{n,u})^{(1+\delta)/4}$. By the above trivial bound on diameters, to show that the probabilities (27) are stretched-exponential as $n \to \infty$, it suffices to see that the probability of the event $\{\mathrm{A}(\mathbf{M}_{n,u}) \neq \emptyset\}$ is stretched-exponential as $n \to \infty$.

By Proposition 7, conditionally on $\mathbf{T}_{n,u}$, each block $\mathfrak{b}_v^{\mathbf{M}_{n,u}}$ is sampled uniformly from 2-connected maps with size $k_v(\mathbf{T}_{n,u})/2$ respectively. Therefore, conditionally on $\mathbf{T}_{n,u}$, for each vertex v in $\mathbf{T}_{n,u}$ we have

$$\begin{aligned} P\big(v \in \mathrm{A}(\mathbf{M}_{n,u}) \,|\, \mathbf{T}_{n,u}\big) &= \mathbb{1}_{\{k_v(\mathbf{T}_{n,u})/2 \geqslant n^{1/6}\}} P\left(\mathrm{diam}(B_{k/2}^{\mathrm{map}}) \geqslant k^{(1+\delta)/4}\right)\Big|_{k=k_v(\mathbf{T}_{n,u})} \\ &\leqslant \sup_{k/2 \geqslant n^{1/6}} P\left(\mathrm{diam}(B_{k/2}^{\mathrm{map}}) \geqslant k^{(1+\delta)/4}\right) \\ &\leqslant \sup_{k \geqslant n^{1/6}} P\left(\mathrm{diam}(B_k^{\mathrm{map}}) \geqslant k^{(1+\delta)/4}\right). \end{aligned}$$

Since $\mathbf{T}_{n,u}$ has $2n+1$ vertices, this yields by a union bound,

$$P(\mathrm{A}(\mathbf{M}_{n,u}) \neq \emptyset) \leqslant (2n+1) \sup_{k \geqslant n^{1/6}/2} P\left(\mathrm{diam}(B_k^{\mathrm{map}}) \geqslant k^{(1+\delta)/4}\right),$$

which is stretched-exponential as $n \to \infty$ by Proposition 10, as announced. A similar use of Proposition 10 proves that the probabilities (28) are stretched-exponential as $n \to \infty$. □

5.2 The supercritical and critical cases

5.2.1 Statement of the result

For $1 < \theta \leqslant 2$, let us denote by $\mathcal{T}^{(\theta)}$ a θ-stable Lévy tree equipped with its mass measure. There are several equivalent constructions of these objects. A common way is to define them *via* excursions of θ-stable Lévy processes. Namely, $\mathcal{T}^{(\theta)}$ is the real tree encoded by the height process of an excursion of length one of a θ-stable Lévy process, see [Duq03]. To fix a normalization for $\mathcal{T}^{(\theta)}$, we consider in the construction an excursion obtained by a cyclic shift from a θ-stable Lévy Bridge with Laplace exponent $\lambda \mapsto \lambda^\theta$. Note that the measured metric space $\mathcal{T}^{(2)}$ corresponds to $\sqrt{2}$ times the *Brownian Continuum Random Tree*, which is encoded by an excursion of length 1 of the standard Brownian motion. The precise definition *via* excursions is not important for our statement and one can take Proposition 14 below as an alternative definition.

Theorem 5. *There exist positive constants $(\kappa_u^{\mathrm{map}}, \kappa_u^{\mathrm{quad}})_{u \geqslant u_C}$ such that we have the following joint convergences in distribution, in the Gromov-Hausdorff-Prokhorov sense:*

1. *If $u > u_C$, we have*

$$\frac{\sigma(u)}{\sqrt{2}}(2n)^{-1/2} \cdot \left(\underline{\mathbf{T}}_{n,u}, \underline{\mathbf{M}}_{n,u}, \underline{\mathbf{Q}}_{n,u}\right) \xrightarrow[n\to\infty]{\mathrm{GHP},(d)} \left(\mathcal{T}^{(2)}, \kappa_u^{\mathrm{map}} \cdot \mathcal{T}^{(2)}, \kappa_u^{\mathrm{quad}} \cdot \mathcal{T}^{(2)}\right),$$

   where we set

$$\sigma(u)^2 = 1 + \frac{4u\,(y(u))^2\,B''(y(u))}{uB(y(u)) + 1 - u} = \frac{3u - 3 + 2\sqrt{u(u-1)}}{5u - 9}. \tag{29}$$

2. *If $u = u_C = 9/5$, we have*

$$\frac{2}{3}(2n)^{-1/3} \cdot \left(\underline{\mathbf{T}}_{n,u_C}, \underline{\mathbf{M}}_{n,u_C}, \underline{\mathbf{Q}}_{n,u_C}\right) \xrightarrow[n\to\infty]{\mathrm{GHP},(d)} \left(\mathcal{T}^{(3/2)}, \kappa_{u_C}^{\mathrm{map}} \cdot \mathcal{T}^{(3/2)}, \kappa_{u_C}^{\mathrm{quad}} \cdot \mathcal{T}^{(3/2)}\right).$$

Additionally, the constants $(\kappa_u^{\mathrm{map}}, \kappa_u^{\mathrm{quad}})_{u \geqslant u_C}$ can be expressed as follows.

$$\kappa_u^{\mathrm{map}} = \sum_{j \geqslant 1} 2j\mu^u(2j)\mathcal{D}_j^{\mathrm{map}} \qquad \text{and} \qquad \kappa_u^{\mathrm{quad}} = \sum_{j \geqslant 1} 2j\mu^u(2j)\mathcal{D}_j^{\mathrm{quad}}, \tag{30}$$

where $\mathcal{D}_j^{\mathrm{map}}$ (resp. $\mathcal{D}_j^{\mathrm{quad}}$) is the expectation of the distance, in a uniform 2-connected map with j edges (resp. simple quadrangulation with j faces) of the distance of the root vertex to the base vertex of a uniform corner (resp. to the closest endpoint of a uniform edge).

Remark 5. *Let us explain how one gets the second equality of (29), which allows to draw Fig. 18. From the proof of the convergence, one gets:*

$$\sigma(u)^2 = 1 + \frac{4u\,(y(u))^2\,B''(y(u))}{uB(y(u)) + 1 - u}.$$

Differentiating the algebraic equation (18) satisfied by B with respect to y and taking (16) as a fourth equation gives a polynomial system from which one can get $B(y(u))$ and $B''(y(u))$ as functions

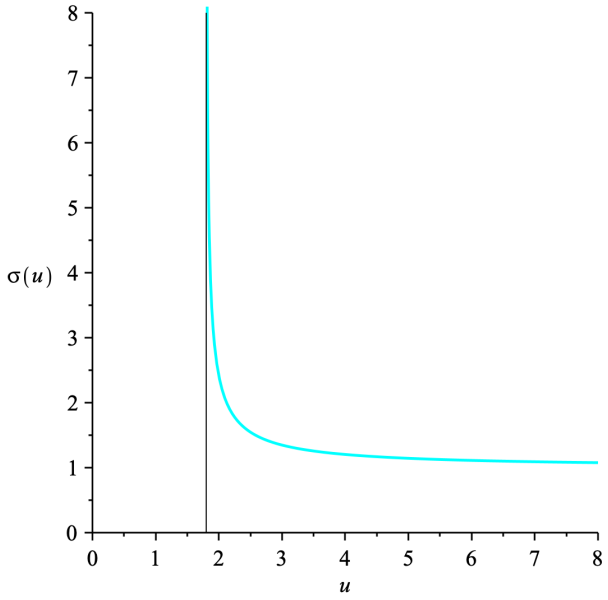

Figure 18: Plot of σ as a function of u. The vertical line corresponds to $u = u_C$.

of u only (using the resultant or a Gröbner basis algorithm). Then, one can conclude from the expression of $\sigma(u)$ in (29).

Remark 6. *The quantity κ_u^{quad} could in principle be obtained via the explicit formula obtained in [BG10] for the generating function g_ℓ of simple edge-rooted quadrangulations with a distinguished edge at prescribed distance ℓ from the root vertex.*

5.2.2 Discussion and overview of the proof

Let $u \geqslant u_C$.

Consider a geodesic in either $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$ between two distant blocks $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$, respectively indexed by v and $\widetilde{v}$ in the block-tree. This geodesic must go through all the blocks whose index w in the block-tree $\mathbf{T}_{n,u}$ is on the path from v to $\widetilde{v}$, in the order induced by this path in the tree.

We have seen in Proposition 7 that under the law of $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$, the blocks are independent conditionally on the block-tree, and when $u \geqslant u_C$ they tend to all have non-macroscopic $o(n)$ size by Theorems 3 and 4. One therefore expects that when n is large, the distance between two distant blocks $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$ falls into a *law of large numbers* behaviour and is of the same order as $d_{\mathbf{T}_{n,u}}(v, \widetilde{v})$.

According to this heuristic, the macroscopic distances in $\mathbf{M}_{n,u}$ and $\mathbf{Q}_{n,u}$ should be concentrated around a deterministic scalar multiple of the distances in $\mathbf{T}_{n,u}$. But $\mathbf{T}_{n,u}$ is a critical Galton-Watson tree conditioned to have $2n+1$ vertices, with explicit tail asymptotic given by Theorem 1 for its offspring distribution, yielding that its scaling limit is a stable tree.

To make this heuristic work, one needs to understand the typical distribution of degrees on a typical path in the tree. It turns out that on a typical path in a size-n critical Galton-Watson tree, the degrees are asymptotically independent and identically distributed; and moreover they are distributed as the size-biased version of the offspring distribution. This will be obtained by a spine

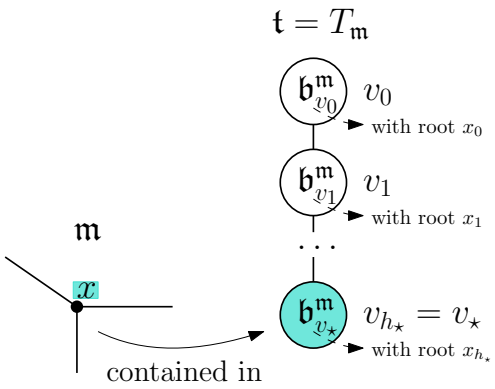

Figure 19: Situation of Lemma 12.

decomposition for trees, adapted to our context.

We bring the attention of the reader to the fact that a proof similar in spirit has been done for the Gromov-Hausdorff metric in the general abstract setup of enriched trees by Stufler [Stu20, Theorem 6.60], and we could readily apply this result to deal with the case $u > u_C$, modulo a technical complication regarding the additivity of distances in the quadrangulation case. When $u = u_C$ however, the distances within blocks have fat tails, so we fall outside the scope of Stufler's result. To deal with this, our last technical ingredient is a suitable large deviation estimate: we show that after an adequate truncation of the variables depending on n, large (and moderate) deviation events still have very small probability.

We now proceed with the proof.

5.2.3 Additivity of the distances along consecutive blocks

In Lemmas 12 and 13, we justify that a macroscopic distance is indeed a sum of distances on "in-between" blocks, in the case of blocks lying on the same branch in the block-tree.

The map case. For $\mathfrak{b}$ a 2-connected map, and l an integer in $\{1, \dots, 2|\mathfrak{b}|\}$, let us denote by $D(\mathfrak{b}, l)$ the graph distance in $\mathfrak{b}$ between its root vertex and the vertex on which lies the l-th corner of $\mathfrak{b}$ in breadth-first order (or in whatever arbitrary ordering rule is chosen in the block-tree decomposition, see Section 2.3).

Fix a vertex x on $\mathfrak{m}$. Let $v_\star$ be the vertex v of the block-tree $\mathfrak{t}$, closest to the root of $\mathfrak{t}$, such that x is a vertex of $\mathfrak{b}^{\mathfrak{m}}_v$. Denote by $h_\star := h_{\mathfrak{t}}(v_\star)$, and by $(v_i)_{0 \leqslant i \leqslant h_\star}$ the ancestor line of $v_\star$ in $\mathfrak{t}$, with v_0 the root and $v_{h_\star} = v_\star$. For $0 \leqslant i \leqslant h_\star$, let x_i be the root vertex of $\mathfrak{b}^{\mathfrak{m}}_{v_i}$. Finally, let $(l_i)_{0 \leqslant i < h_\star}$ be the respective breadth-first index of the corner in $\mathfrak{b}^{\mathfrak{m}}_{v_i}$ in which the block $\mathfrak{b}^{\mathfrak{m}}_{v_{i+1}}$ is attached. The situation is illustrated on Fig. 19.

Lemma 12. *For $0 \leqslant i \leqslant h_\star$, we have*

$$d_{\mathfrak{m}}(x, x_i) = d_{\mathfrak{b}^{\mathfrak{m}}_{v_\star}}(x, x_{h_\star}) + \sum_{i \leqslant j < h_\star} D(\mathfrak{b}^{\mathfrak{m}}_{v_j}, l_j).$$

Proof. By definition, $D(\mathfrak{b}^{\mathfrak{m}}_{v_j}, l_j) = d_{\mathfrak{m}}(x_j, x_{j+1})$ for $i \leqslant j < h_\star$. We get by the triangle inequality that the left-hand-side is at most the right-hand side. Therefore it suffices to show that any geodesic path in $\mathfrak{m}$ from x to x_i visits each of the points $(x_j)_{i<j\leqslant h_\star}$, in decreasing order of j.

Let j be such that $i < j \leqslant h_\star$. Denote by $\mathfrak{t}_j$ the tree of descendants of v_j in $\mathfrak{t}$ (rooted in v_j) and also $\mathfrak{m}_j$ and $\widetilde{\mathfrak{m}}_j$ the submaps of $\mathfrak{m}$ made of the blocks $(b^{\mathfrak{m}}_v)_{v\in\mathfrak{t}_j}$ and $(b^{\mathfrak{m}}_v)_{v\in\mathfrak{t}\setminus\mathfrak{t}_j}$ respectively. By the recursive description of the block-tree, the submaps $\mathfrak{m}_j$ and $\widetilde{\mathfrak{m}}_j$ share only the vertex x_j. But x is a vertex of $\mathfrak{m}_j$ since $v_\star$ is a descendant of v_j, and x_i is a vertex of $\widetilde{\mathfrak{m}}_j$ since v_i is an ancestor of v_j. Hence any injective path between x and x_i must visit x_j in decreasing order of j; and in particular for a geodesic path.

Notice that it does not require the (x_j) to be mutually distinct. This concludes the proof. □

The quadrangulation case. A slight complication arises for quadrangulations because the "interface" between two blocks is a double edge, containing two vertices instead of a single vertex in the map case. At first sight it is thus unclear through which of these vertices a geodesic should go. We show that there is a canonical choice: the vertex between those two which is closest to the root vertex. This relies crucially on the fact that quadrangulations are bipartite.

Fix a quadragulation $\mathfrak{q}$. For $\mathfrak{b}$ a simple block of $\mathfrak{q}$, and l an integer in $\{1, \dots, 2|\mathfrak{b}|\}$, let us denote by $D_{\mathfrak{q}}(\mathfrak{b}, l)$ the graph distance in $\mathfrak{b}$ between the endpoint of the l-th edge of $\mathfrak{b}$ in the ordering described thereafter, and the endpoint of the root edge of $\mathfrak{b}$ which is closest to the root vertex of $\mathfrak{q}$. The order on the edges of $\mathfrak{b}$ that we use is the image of the lexicographic order on vertices of $\mathfrak{t}$ *via* the block-tree decomposition. This is consistent with the ordering of corners in the map case.

Fix a vertex x on $\mathfrak{q}$. Similarly to the map case, let $v_\star$ be the vertex v of the block-tree $\mathfrak{t}$, closest to the root of $\mathfrak{t}$, such that x is a vertex of $\mathfrak{b}^{\mathfrak{q}}_v$. Define accordingly $h_\star = h_{\mathfrak{t}}(v_\star)$ and the ancestor line $(v_i)_{0\leqslant i\leqslant h_\star}$ of $v_\star$ in the block-tree $\mathfrak{t}$, with v_0 the root and $v_{h_\star} = v_\star$. Let also $(x_i)_{0\leqslant i\leqslant h_\star}$ be the respective root vertex of $\mathfrak{b}_{v_i}$. Finally, let $(l_i)_{0\leqslant i<h_\star}$ be the respective breadth-first index of the edge in $\mathfrak{b}^{\mathfrak{q}}_{v_i}$ to which the root edge of $\mathfrak{b}^{\mathfrak{q}}_{v_{i+1}}$ is attached.

Lemma 13. *For all $0 \leqslant i \leqslant h_\star$, there exists $\delta_{x,x_i} \in \{0, \pm 1, \pm 2\}$ such that*

$$d_{\mathfrak{q}}(x, x_i) = \delta_{x,x_i} + d_{\mathfrak{b}^{\mathfrak{q}}_{v_\star}}(x, x_{h\star}) + \sum_{i\leqslant j<h_\star} D_{\mathfrak{q}}(\mathfrak{b}^{\mathfrak{q}}_{v_j}, l_j).$$

Proof. The idea is quite similar in principle as in the preceding lemma, except that consecutive blocks share two vertices in the quadrangulation case, instead of one.

For $0 \leqslant j \leqslant h_\star$, denote by y_j the endpoint of the root-edge of $\mathfrak{b}^{\mathfrak{q}}_{v_j}$ which is closest to the root vertex of $\mathfrak{q}$. In particular y_0 is the root vertex of $\mathfrak{q}$. Let $0 \leqslant i \leqslant h_\star$. Then, by construction, y_i and x_i are adjacent to the root edge of $\mathfrak{b}^{\mathfrak{q}}_{v_i}$. Notice that a geodesic from x to x_i must visit at least one of the endpoints of the root-edge of $\mathfrak{b}_{v_\star}$, which is at distance 0 or 1 of $y_{h_\star}$, and that x_i and y_i are at distance 0 or 1. Therefore, there exists some $\delta_{x,x_i} \in \{0, \pm 1, \pm 2\}$ such that

$$d_{\mathfrak{q}}(x, x_i) = \delta_{x,x_i} + d_{\mathfrak{b}^{\mathfrak{q}}_{v_\star}}(x, x_{h\star}) + d_{\mathfrak{q}}(y_{h_\star}, y_i).$$

We shall prove the following, which are sufficient to conclude:

1. For $0 \leqslant i \leqslant h_\star$, it holds that $d_{\mathfrak{q}}(y_{h_\star}, y_i) = \sum_{i\leqslant j<h_\star} d_{\mathfrak{q}}(y_{j+1}, y_j)$;
2. For $0 \leqslant j < h_\star$, it holds that $d_{\mathfrak{q}}(y_{j+1}, y_j) = D_{\mathfrak{q}}(\mathfrak{b}^{\mathfrak{q}}_{v_j}, l_j)$.

It is even sufficient to show the following:

$$\forall 0 \leqslant i \leqslant j \leqslant k \leqslant h_\star, \quad d_{\mathfrak{q}}(y_i, y_k) = d_{\mathfrak{q}}(y_i, y_j) + d_{\mathfrak{q}}(y_j, y_k). \tag{31}$$

Indeed, assuming (31) holds, by applying it iteratively, we directly get that $d_{\mathfrak{q}}(y_{h_\star}, y_i) = \sum_{i \leqslant j < h_\star} d_{\mathfrak{q}}(y_{j+1}, y_j)$. To verify the second set of identities, recall that y_j is defined as the endpoint of the root edge of $b^{\mathfrak{q}}_{v_j}$ which is closest to the root vertex y_0 of $\mathfrak{q}$. Denote by y'_j the other endpoint. Then, for $0 \leqslant j < h_\star$ we have

$$D_{\mathfrak{q}}(\mathfrak{b}^{\mathfrak{q}}_{v_j}, l_j) = \min\left(d_{\mathfrak{b}^{\mathfrak{q}}_{v_j}}(y_{j+1}, y_j), d_{\mathfrak{b}^{\mathfrak{q}}_{v_j}}(y'_{j+1}, y_j)\right) = \min\left(d_{\mathfrak{q}}(y_{j+1}, y_j), d_{\mathfrak{q}}(y'_{j+1}, y_j)\right).$$

The first equality comes from the definition of $D_{\mathfrak{q}}(\mathfrak{b}^{\mathfrak{q}}_{v_j}, l_j)$, and the second one from the fact that within a block $\mathfrak{b}$ of $\mathfrak{q}$, the graph distance respective to $\mathfrak{q}$ and the graph distance respective to $\mathfrak{b}$ coincide. Then, assuming (31), it holds that

$$d_{\mathfrak{q}}(y_{j+1}, y_j) = d_{\mathfrak{q}}(y_{j+1}, y_0) - d_{\mathfrak{q}}(y_j, y_0) \leqslant d_{\mathfrak{q}}(y'_{j+1}, y_0) - d_{\mathfrak{q}}(y_j, y_0) \leqslant d_{\mathfrak{q}}(y'_{j+1}, y_j),$$

where the first inequality comes from the definition of y_{j+1}, and the second inequality from triangle inequality. In particular, the above minimum is $d_{\mathfrak{q}}(y_{j+1}, y_j)$ and we have, as needed, $d_{\mathfrak{q}}(y_{j+1}, y_j) = D_{\mathfrak{q}}(\mathfrak{b}^{\mathfrak{q}}_{v_j}, l_j)$.

It still remains to prove (31). Let us first prove the case $i = 0$ and then deduce the general case. Let $0 \leqslant j \leqslant k \leqslant h_\star$, and let γ be a geodesic path from y_0 to y_k. If γ visits y_j, we readily have

$$d_{\mathfrak{q}}(y_0, y_k) = d_{\mathfrak{q}}(y_0, y_j) + d_{\mathfrak{q}}(y_j, y_k). \tag{32}$$

Otherwise it visits y'_j, and denote by γ_1, γ_2 the portions of γ form y_0 to y'_j, and from y'_j to y_k respectively. By definition of y_j, we have $d_{\mathfrak{q}}(y_0, y_j) \leqslant d_{\mathfrak{q}}(y_0, y'_j)$. But since $\mathfrak{q}$ is a quadrangulation, it is bipartite and the inequality is strict $d_{\mathfrak{q}}(y_0, y_j) < d_{\mathfrak{q}}(y_0, y'_j)$. Form $\widetilde{\gamma}_1$ the concatenation of a geodesic path from y_0 to y_j and of the oriented edge (y_j, y'_j). Then, from the strict inequality we mentioned, $\mathrm{len}(\widetilde{\gamma}_1) \leqslant \mathrm{len}(\gamma_1)$, and in particular the concatenation of $\widetilde{\gamma}_1$ and γ_2 is a geodesic path from y_0 to y_k which visits y_j. Therefore, the identity (32) also holds.

Finally, let us deduce the case $i \neq 0$. Let $0 \leqslant i \leqslant j \leqslant k \leqslant h_\star$. We have

$$\begin{aligned} d_{\mathfrak{q}}(y_i, y_k) = d_q(y_0, y_k) - d_{\mathfrak{q}}(y_0, y_i) &= (d_q(y_0, y_k) - d_{\mathfrak{q}}(y_0, y_j)) + (d_q(y_0, y_j) - d_{\mathfrak{q}}(y_0, y_i)) \\ &= d_{\mathfrak{q}}(y_j, y_k) + d_{\mathfrak{q}}(y_i, y_j). \end{aligned}$$

This proves (31) and concludes the proof of Lemma 13.

□

5.2.4 Scaling limit and largest degree of critical Galton-Watson trees

A slight technical complication that arises in our setting is that the block-tree has a *lattice* offspring distribution with span 2, in the sense of the following definition

Definition 9. *A measure μ on $\mathbb{Z}$ is called* lattice *if its support is included in a subset $b + d\mathbb{Z}$ of $\mathbb{Z}$, with $d \geqslant 2$. The largest such d is called its* span. *If $d = 1$, μ is called* non-lattice.

The results that we need [Kor13, Theorem 3] are stated for non-lattice offspring distributions. This turns out to be purely for convenience and we state the following more general result that is suited to our needs.

We recall that a probability distribution μ with mean m_μ is said to be in the domain of attraction of a stable law of index $\theta \in (1,2]$ if there exist positive constants $(C_n)_{n\geqslant 0}$ such that we have the following convergence in distribution

$$\frac{U_1 + \cdots + U_n - nm_\mu}{C_n} \xrightarrow[n\to\infty]{(d)} X^{(\theta)}, \tag{33}$$

where $(U_1, \dots U_n)$ are i.i.d. samples of the law μ, and $X^{(\theta)}$ is a random variable with Laplace transform $E\left[\exp(-\lambda X^{(\theta)})\right] = \exp(\lambda^\theta)$.

Proposition 14. *For all $1 < \theta \leqslant 2$, there exists a random measured metric space $\mathcal{T}^{(\theta)} = \left(\mathcal{T}^{(\theta)}, d^{(\theta)}, \nu^{(\theta)}\right)$ satisfying the following scaling limit result.*

Let μ be a probability distribution on $\mathbb{Z}_{\geqslant 0}$, with $\mu(1) \neq 1$, and which is assumed to be critical. Assume additionally that it is in the domain of attraction of a stable law of index $\theta \in (1,2]$. Let $d \geqslant 1$ be the span of the measure μ. Then under those assumptions, we have

1. *For all m large enough, the $\mathrm{GW}_\mu(\mathrm{d}T)$-probability that T has dm edges is positive. This probability is equivalent to $c_\theta/(C_{dm}dm)$ for some constant $c_\theta > 0$.*

2. *If we denote by T_n a GW_μ-tree conditioned to have n edges, then*

$$\left(\frac{C_{dm}}{dm}\right) \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{(d)} \mathcal{T}^{(\theta)},$$

   in the Gromov-Hausdorff-Prokhorov *sense, with $(C_n)_{n\geqslant 0}$ the sequence in (33).*

3. *The largest degree in T_{dm} is of order at most C_{dm}, in the sense that for any $\varepsilon > 0$*

$$P\left(\exists v \in T_{dm}, k_v(T_{dm}) \geqslant (C_{dm})^{1+\varepsilon}\right) \xrightarrow[m\to\infty]{} 0.$$

Proof. The first statement can be obtained by a straightforward adaptation of the proof of [Kor13, Lemma 1], which relies on a local limit theorem and the cycle lemma. We specify below how this local limit theorem should be adapted. The cycle lemma adapts straightforwardly.

For the second statement, let us justify that [Kor13, Theorem 3] still applies when the *non-lattice* (or *aperiodic*) assumption is dropped, but with the number of vertices $n+1$ taken only along the subsequence $(dm+1)_{m\geqslant 0}$. This will prove functional convergence of the contour functions of the trees $(T_{dm})_m$ when properly rescaled, to the contour function of $\mathcal{T}^{(\theta)}$. This convergence of contour functions is sufficient to get the announced Gromov-Hausdorff-Prokhorov convergence.

The local limit theorem [Kor13, Theorem 2, (ii)] changes as follows

$$\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| \frac{a_n}{d} P\left(Y_n = k\right) - p_1\left(\frac{k}{a_n}\right) \right| = 0.$$

See for instance [Ibr71, Theorem 4.2.1]. Notice that the only difference with the *non-lattice* ($d = 1$) local limit theorem is the factor $1/d$ in the last display. Examining the details of Kortchemski's

arguments, this extra $1/d$ factor would appear only in the discrete absolute continuity relations which are used in the proof. But in each instance, it would appear in both the numerator and denominator of some fraction. Hence the fraction simplifies and this factor has no impact on the proof, which carries without change, except that the integer n, which in the paper is the number of *vertices*, should now only be taken in $d\mathbb{Z}+1$.

Finally, in order to get the third statement, one can take as a basis the local limit theorem above. From this, one can get the functional convergence of the Łucasiewicz path of T_{dm}, when it is rescaled by dm in time and C_{dm} in space. In particular, $(C_{dm})^{-1}$ times the largest degree in T_{dm} is tight, and one obtains the claimed probabilistic bound. One could for instance use the same arguments as in the proof of [KM21, Proposition 3.4] □

Corollary 15 then just identifies the explicit scaling constants in specific instances of the above-mentioned scaling limit theorem.

Corollary 15. *Let μ be a critical probability distribution on $\mathbb{Z}_{\geqslant 0}$ with span $d \geqslant 1$, and with $\mu(1) \neq 1$. Denote by T_n a GW_μ-tree conditioned to have n edges, for $n \in d\mathbb{Z}$ large enough. Then the following holds.*

1. *If μ has finite variance σ^2, then $P(|T| = dm) \sim cm^{-3/2}$ for some constant $c > 0$, and*

$$(dm)^{-1/2} \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{\mathrm{GHP},(d)} \frac{\sqrt{2}}{\sigma} \cdot \mathcal{T}^{(2)}.$$

   Additionally for all $\varepsilon > 0$ the largest degree of T_{dm} is $o(m^{1/2+\varepsilon})$ in probability.

2. *If $\mu([x,+\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1,2)$, then $P(|T| = dm) \sim c'_\theta m^{-(1+1/\theta)}$ for some constant $c'_\theta > 0$, and*

$$(dm)^{-(1-1/\theta)} \cdot \underline{T}_{dm} \xrightarrow[m\to\infty]{\mathrm{GHP},(d)} \left[\frac{\theta-1}{c\,\Gamma(2-\theta)}\right]^{1/\theta} \cdot \mathcal{T}^{(\theta)}.$$

   Additionally for all $\varepsilon > 0$ the largest degree of T_{dm} is $o(m^{1/\theta+\varepsilon})$ in probability.

Proof. Note that in the case where ν has finite exponential moments, [MM03] treats the case of lattice distributions. That would suffice for our applications when $u > u_C$. We still need the second statement to treat the case $u = u_C$. Let us apply the preceding proposition and identify the right constants, in these two cases.

Statement 1. If μ has finite variance σ^2, then by the Central Limit Theorem, for i.i.d. samples $(U_i)_i$ of the law μ, we have the convergence in distribution

$$\frac{U_1 + \cdots + U_n - n}{\sigma \cdot n^{1/2}} \xrightarrow[n\to\infty]{(d)} \mathcal{G},$$

where $\mathcal{G}$ is a standard normal variable. In particular, $\mathcal{G}$ has the same law as $\frac{1}{\sqrt{2}} \cdot X^{(2)}$. Therefore, the hypotheses of Proposition 14 are satisfied, with

$$C_n = \tfrac{\sigma}{\sqrt{2}} \cdot n^{1/2},$$

and the conclusion follows from this proposition.

Statement 2. We consider the case where $\mu([x,+\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ with $\theta \in (1,2)$ and $c>0$. Let $U := U_1$ and let us also introduce the notation

$$\begin{aligned} M_1(x) &= \int_x^\infty \mu(\mathrm{d}y) = \mu([x,+\infty)) \\ M_2(x) &= \int_x^\infty M_1(y)\,\mathrm{d}y \\ M_3(x) &= \int_0^x M_2(y)\mathrm{d}y. \end{aligned}$$

The function M_3 is non-decreasing and using the assumed tail asymptotic of μ, one has the asymptotic $M_3(x) \sim cx^{2-\theta}/(2-\theta)(\theta-1)$. We may therefore use the Karamata Tauberian theorem [BGT89, Theorem 1.7.1] to get

$$\widehat{M_3}(h) \sim \frac{c\,\Gamma(3-\theta)}{(2-\theta)(\theta-1)} h^{\theta-2} = \frac{c\,\Gamma(2-\theta)}{\theta-1} h^{\theta-2},$$

where $\widehat{M_3}$ is the *Laplace-Stieltjes transform* of M_3, defined — *e.g.* in [BGT89, Paragraph 1.7.0b] — as

$$\widehat{M_3}(h) = h\int_0^\infty e^{-hx} M_3(x)dx$$

for all h for which the integral converges absolutely. Then, if we integrate by parts three times, we obtain

$$\begin{aligned} E\left[\exp(-h(U-1))\right] = \int_0^\infty \mathrm{e}^{-h(x-1)}\,\mu(\mathrm{d}x) &= \mathrm{e}^h - h\mathrm{e}^h M_2(0) + h^3\mathrm{e}^h \int_0^\infty \mathrm{e}^{-hx} M_3(x)\,\mathrm{d}x \\ &= \mathrm{e}^h - h\mathrm{e}^h M_2(0) + h^2\mathrm{e}^h\,\widehat{M_3}(h), \end{aligned}$$

This, together with the fact that $M_2(0)=1$ since it is the expectation of μ, yields the following expansion when $h \to 0^+$,

$$E\left[\exp(-h(U-1))\right] = 1 + \frac{c\,\Gamma(2-\theta)}{\theta-1}\cdot h^\theta\,(1+o(1))\,. \tag{34}$$

Now, if we set

$$C_n = \left(\frac{c\,\Gamma(2-\theta)}{\theta-1}\right)^{1/\theta} n^{1/\theta},$$

and plug $h = \lambda/C_n$ into (34), we get for all $\lambda \geqslant 0$,

$$E\left[\exp\left(-\lambda\,\tfrac{U_1+\cdots+U_n-n}{C_n}\right)\right] = \left(E\left[\exp(-\tfrac{\lambda}{C_n}(U-1))\right]\right)^n \xrightarrow[n\to\infty]{} \exp\left(\lambda^\theta\right).$$

Hence there is convergence in distribution of $\frac{U_1+\cdots+U_n-n}{C_n}$ to $X^{(\theta)}$, as required in Proposition 14. So this proposition applies with the above-chosen sequence $(C_n)_n$, and the conclusion follows. □

5.2.5 Scaling limit of critical Galton-Watson trees equipped with a random measure

We shall need a version of the GHP scaling limits in Corollary 15, when the trees $\mathfrak{t}$ under consideration are equipped with some random measure on their vertices, instead of the uniform measure $\nu_{\mathfrak{t}}$. Let us describe more specifically our setting.

Let μ be a probability measure on $\mathbb{Z}_{\geqslant 0}$ and $\eta = (\eta_k)_{k\geqslant 0}$ be a family of Borel probability measures on $\mathbb{R}_{\geqslant 0}$. We shall define an enriched version $\mathcal{L}(\mu, \eta)$ of the Galton-Watson law $\mathrm{GW}(\mu)$, defined on the set of pairs $(\mathfrak{t}, f)$ such that $\mathfrak{t}$ is a tree and f is a non-negative function $f\colon V(\mathfrak{t}) \to \mathbb{R}_{\geqslant 0}$. Namely, to sample a random pair $(\mathbf{T}, \mathbf{f})$ with law $\mathcal{L}(\mu, \eta)$ first sample $\mathbf{T}$ according to $\mathrm{GW}(\mu)$, and then sample conditionally on $\mathbf{T}$ the variables $\mathbf{f}(v)$ for $v \in \mathbf{T}$, independently of each other, according to the laws $\eta_{k_v(\mathbf{T})}(\mathrm{d}x)$ respectively.

In particular, the random non-negative function $\mathbf{f}$ defines a random measure on $V(\mathbf{T})$ assigning weight $\mathbf{f}(v)$ to the vertex v. We shall use the same notation $\mathbf{f}$ for this measure, and denote by $|\mathbf{f}|$ its total weight.

Proposition 16. *Let μ be a critical offspring distribution with span $d \geqslant 1$ such that $\mu(1) \neq 1$. Let also $(\eta_k)_{k\geqslant 0}$ be Borel probability laws that are supported on $\mathbb{R}_{\geqslant 0}$. For $n \in d\mathbb{Z}$ large enough, denote by $(T_n, \mathbf{f}_n)$ a sample of the law $\mathcal{L}(\mu, \eta)$ conditioned to the event $\{|\mathbf{T}| = n\}$. If the annealed measure $\sum_k \mu(k)\eta_k(\mathrm{d}s)$ admits a positive and finite first moment, then the following holds.*

1. *If μ has finite variance σ^2, then*

$$(dm)^{-1/2} \cdot \left(V(T_{dm}), d_{T_{dm}}, \frac{\mathbf{f}_{dm}}{|\mathbf{f}_{dm}|} \right) \xrightarrow[m\to\infty]{\mathrm{GHP},(d)} \frac{\sqrt{2}}{\sigma} \cdot \mathcal{T}^{(2)}.$$

2. *If $\mu([x, +\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1, 2)$, then*

$$(dm)^{-(1-1/\theta)} \cdot \left(V(T_{dm}), d_{T_{dm}}, \frac{\mathbf{f}_{dm}}{|\mathbf{f}_{dm}|} \right) \xrightarrow[m\to\infty]{\mathrm{GHP},(d)} \left[\frac{\theta - 1}{c\,\Gamma(2-\theta)} \right]^{1/\theta} \cdot \mathcal{T}^{(\theta)}.$$

We shall first prove a rather general functional law of large numbers for the cumulative sum $s \mapsto \sum_{i\leqslant sn} \mathbf{f}_n(v_i)$, where $(v_i)_i$ are the vertices of T_n listed in depth-first order.

Lemma 17. *Let μ be a critical offspring distribution with span $d \geqslant 1$, and with $\mu(1) \neq 1$. Let also $(\eta_k)_{k\geqslant 0}$ be Borel probability laws on $\mathbb{R}_{\geqslant 0}$. For $n \in d\mathbb{Z}$ large enough, denote by $(T_n, \mathbf{f}_n)$ a sample of the law $\mathcal{L}(\mu, \eta)$ conditioned to the event $\{|\mathbf{T}| = n\}$. Assume that the annealed measure $\sum_k \mu(k)\eta_k(\mathrm{d}s)$ admits a positive and finite first moment and denote by $\overline{\eta} > 0$ its expectation. Assume also that μ is in the domain of attraction of a stable distribution of index α with $1 < \alpha \leqslant 2$. Then there holds the following convergence in probability*

$$\sup_{s\in[0,1]} \left| \frac{1}{n} \sum_{i\leqslant sn} \mathbf{f}_n(v_i) - s\overline{\eta} \right| \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0,$$

where $(v_0, \ldots, v_n)$ are the vertices of T_n listed in depth-first order.

Proof. Let us denote by $(\widetilde{v}_i)_{0\leqslant i\leqslant n}$ a uniform cyclic shift of the sequence $(v_i)_{0\leqslant i\leqslant n}$, that is to say

$\widetilde{v}_i = v_{i+\tau_n \bmod (n+1)}$ for all $0 \leqslant i \leqslant n$, where τ_n is a uniformly random element of $\{0, \dots, n\}$, sampled independently from other variables. Then an elementary re-arranging of sums yields that

$$\sup_{s\in[0,1]} \left| \frac{1}{n} \sum_{i \leqslant sn} \mathbf{f}_n(v_i) - s\overline{\eta} \right| \leqslant 2 \sup_{0\leqslant s\leqslant t\leqslant 1} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right|.$$

Distinguishing upon whether s and t are smaller than $1/2$, and cutting the sum at $1/2$ in the case $s < 1/2 < t$, we can bound further

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t\leqslant 1} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right| \\
&\leqslant 2 \sup_{0\leqslant s\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right| + 2 \sup_{1/2\leqslant s\leqslant t\leqslant 1} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right| \\
&= 2 \sup_{0\leqslant s\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - (t-s)\overline{\eta} \right| + 2 \sup_{0\leqslant s\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{sn\leqslant i\leqslant tn} \mathbf{f}_n(\widetilde{v}_{i+\lceil n/2\rceil}) - (t-s)\overline{\eta} \right| \\
&\leqslant 4 \sup_{0\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - t\overline{\eta} \right| + 4 \sup_{0\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{i\leqslant tn} \mathbf{f}_n(\widetilde{v}_{i+\lceil n/2\rceil}) - t\overline{\eta} \right|.
\end{aligned}$$

Now notice that $(\widetilde{v}_{i+\lceil n/2\rceil})_{0\leqslant i\leqslant n}$ is itself a uniform cyclic shift of the sequence $(v_i)_{0\leqslant i\leqslant n}$, so that the second term in the last display has the same law as the first one, and we only need to bound this one. We have reduced the problem to showing that the following convergence in probability holds

$$\sup_{0\leqslant t\leqslant 1/2} \left| \frac{1}{n} \sum_{i\leqslant tn} \mathbf{f}_n(\widetilde{v}_i) - t\overline{\eta} \right| \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0. \tag{35}$$

We now appeal to the so-called cycle lemma, see [Pit06, Paragraph 6.1] and more precisely Lemma 6.1 for the cycle lemma and Lemma 6.3 for its application to trees. In our setting it implies that the cyclically shifted sequence of degrees $(k_{\widetilde{v}_i}(T_n))_{0\leqslant i\leqslant n}$ has the same law as that of an i.i.d. sequence $(\xi_i)_{0\leqslant i\leqslant n}$ of samples of the law μ conditioned to the event $\{\sum_{0,\leqslant i\leqslant n}(\xi_i - 1) = -1\}$. Now recall that conditionally on T_n, each variable $\mathbf{f}(v_i)$ is sampled according to the law $\eta_{k_{v_i}(T_n)}$ and independently of the family $(\mathbf{f}(v_j))_{j\neq i}$. Therefore the identity in distribution obtained from the cycle lemma admits a straightforward generalization for the cyclically shifted sequence $\big(k_{\widetilde{v}_i}(T_n), \mathbf{f}_n(\widetilde{v}_i)\big)_{0\leqslant i\leqslant n}$. More precisely, let $(\xi_i, X_i)_{i\geqslant 0}$ be an i.i.d. sequence such that ξ_0 has law μ, and such that conditionally on ξ_0 the variable X_0 has law η_{ξ_0}. Then, there holds the following identity in distribution

$$\mathrm{Law}\Big(\big(k_{\widetilde{v}_i}(T_n), \mathbf{f}_n(\widetilde{v}_i)\big)_{0\leqslant i\leqslant n}; P\Big) = \mathrm{Law}\Big(\big(\xi_i, X_i\big)_{0\leqslant i\leqslant n}; P\left(\cdot \;\middle|\; \textstyle\sum_{0\leqslant i\leqslant n}(\xi_i - 1) = -1\right)\Big).$$

Using the Markov property at time $\lfloor n/2\rfloor$ for the random walk $(\sum_{0\leqslant i\leqslant k}(\xi_i - 1))_{k\geqslant 0}$, we get for every non-negative Borel function $F\colon (\mathbb{Z}\times\mathbb{R})^{\lfloor n/2\rfloor+1} \to \mathbb{R}_{\geqslant 0}$ the following

$$\begin{aligned}
E\left[F\Big(\big(k_{\widetilde{v}_i}(T_n), \mathbf{f}_n(\widetilde{v}_i)\big)_{0\leqslant i\leqslant n/2}\Big)\right] &= E\left[F\Big(\big(\xi_i, X_i\big)_{0\leqslant i\leqslant n/2}\Big) \frac{\mathbb{1}\big\{\sum_{0\leqslant i\leqslant n}(\xi_i - 1) = -1\big\}}{P\left(\sum_{0\leqslant i\leqslant n}(\xi_i - 1) = -1\right)}\right] \\
&= E\left[F\Big(\big(\xi_i, X_i\big)_{0\leqslant i\leqslant n/2}\Big) \frac{q_{n-\lfloor n/2\rfloor}(-1 - \sum_{0\leqslant i\leqslant n/2}(\xi_i - 1))}{q_{n+1}(-1)}\right],
\end{aligned}$$

where we used the notation $q_k(j) = P\big(\sum_{1\leqslant i\leqslant k}(\xi_i - 1) = j\big)$. Let us remark that there exists n_0 such that $q_n(-1) \neq 0$ for all the integers $n \geqslant n_0$ which belong to $d\mathbb{Z}$, and that

$$\sup_{n\geqslant n_0,\, n\in d\mathbb{Z}} \sup_{j\in\mathbb{Z}} \frac{q_{n-\lfloor n/2\rfloor}(l)}{q_{n+1}(-1)} < +\infty. \tag{36}$$

Indeed, we may use the local limit theorem [Ibr71, Theorem 4.2.1] which covers the case of random walks on $\mathbb{Z}$ whose increments have law a (possibly non-aperiodic) distribution in the domain of attraction of a stable distribution with index $\alpha \in (1, 2]$, such as the random walk $\big(\sum_{0\leqslant i\leqslant k}(\xi_i - 1)\big)_{k\geqslant 0}$. This gives us

$$\lim_{k\to\infty} \sup_{j\in\mathbb{Z}} \left| \frac{B_k}{d} q_k(j) - g\left(\frac{j}{B_k}\right)\right| = 0,$$

where g is the density function of some stable distribution with index α satisfying notably $g(0) \neq 0$, and where $(B_k)_k$ is a sequence of numbers such that $(k^{-1/\alpha}B_k)_k$ is slowly varying by [Ibr71, Paragraph 2.2]. We easily deduce (36) from the last display. Therefore there exists a constant $C > 0$ such that for every non-negative Borel function $F\colon (\mathbb{Z}\times\mathbb{R})^{\lfloor n/2\rfloor+1} \to \mathbb{R}_{\geqslant 0}$, we have for $n \geqslant n_0$,

$$E\left[F\Big(\big(k_{\widetilde{v}_i}(T_n), \mathbf{f}_n(\widetilde{v}_i)\big)_{0\leqslant i\leqslant n/2}\Big)\right] \leqslant C \cdot E\left[F\Big(\big(\xi_i, X_i\big)_{0\leqslant i\leqslant n/2}\Big)\right].$$

We deduce for every $\varepsilon > 0$ and every $n \geqslant n_0$,

$$P\left(\sup_{0\leqslant t\leqslant 1/2}\left|\frac{1}{n}\sum_{i\leqslant tn}\mathbf{f}_n(\widetilde{v}_i) - t\overline{\eta}\right| \geqslant \varepsilon\right) \quad \leqslant \quad C \quad \cdot \quad P\left(\sup_{0\leqslant t\leqslant 1/2}\left|\frac{1}{n}\sum_{i\leqslant tn} X_i - t\overline{\eta}\right| \geqslant \varepsilon\right). \tag{37}$$

Notice that the variables $(X_i)_{i\geqslant 0}$ are i.i.d. with mean $\overline{\eta}$ by definition. By the strong law of large numbers, it holds almost surely that for all $t \in [0, 1] \cup \mathbb{Q}$,

$$\frac{1}{N}\sum_{i\leqslant tN} X_i \xrightarrow[N\to\infty]{} t\overline{\eta}.$$

Since the variables (X_i) are non-negative, the left-hand-side is a (random) non-decreasing function of t for all $N \geqslant 1$. In particular, the pointwise almost sure convergence above yields by Dini's theorem almost sure convergence in the sup norm, namely

$$\sup_{0\leqslant t\leqslant 1/2}\left|\frac{1}{n}\sum_{i\leqslant tn} X_i - t\overline{\eta}\right| \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{\text{a.s.}} 0.$$

Combining this with (37), we obtain the desired convergence in probability (35) and this concludes the proof. □

Proof of Proposition 16. Let U be sampled uniformly and independently of other variables and for $n \in d\mathbb{Z}$ large enough, let $(v_0, \ldots, v_n)$ be the vertices of T_n listed in depth-first order. We denote by $x_n(U)$ the vertex $v_{\lfloor (n+1)U\rfloor}$. Let also $y_n(U)$ be the vertex $v_{k_n(U)}$, where $k_n(U)$ is the smallest index $k \in \{0, \ldots, n\}$ such that $\sum_{i\leqslant k}\mathbf{f}_n(v_i) \geqslant U|\mathbf{f}_n|$. By construction, conditionally on $(T_n, \mathbf{f}_n)$, the random vertex $x_n(U)$ has law ν_{T_n} and $y_n(U)$ has law $\mathbf{f}_n/|\mathbf{f}_n|$. Now by Lemma 17, the sequence of functions $(s \mapsto \frac{1}{|\mathbf{f}_n|}\sum_{i\leqslant sn}\mathbf{f}_n(v_i))_n$ converges in probability for the uniform norm to the identity function $s \mapsto s$

when n tends to ∞ in $d\mathbb{Z}$. We deduce using the definition of $k_n(U)$ that $\left|\frac{k_n(U)}{n} - U\right|$ converges to 0 in probability, and in particular that the same goes for $\left|\frac{k_n(U)}{n+1} - \frac{\lfloor (n+1)U \rfloor}{n+1}\right|$.

Let $\theta = 2$ if μ has finite variance as in case 1. of the statement, or let θ be such that $\mu([x,+\infty)) \underset{x\to\infty}{\sim} cx^{-\theta}$ for some $c > 0$ and $\theta \in (1,2)$ as in case 2. of the statement. Let us set $D_n = n^{-(\theta-1)/\theta}\, d_{T_n}$ the rescaled distance function on $V(T_n)$ and $h_n \colon s \in [0,1] \mapsto n^{-(\theta-1)/\theta}\, h_{T_n}(v_{\lfloor (n+1)s \rfloor})$ be the rescaled height process of T_n. Using the following well-known bound on distances in a tree

$$d_{T_n}(v_i, v_k) \leqslant h_{T_n}(v_i) + h_{T_n}(v_k) - 2 \inf_{j \in \{i,\dots,k\}} h_{T_n}(v_j) + 2,$$

we get the bound

$$\begin{aligned} D_n(x_n(U), y_n(U)) &= \frac{1}{n^{(\theta-1)/\theta}}\, d_{T_n}(v_{k_n(U)}, v_{\lfloor (n+1)U \rfloor}) \\ &\leqslant 2\, \omega_{h_n}\left(\left|\tfrac{k_n(U)}{n+1} - \tfrac{\lfloor (n+1)U \rfloor}{n+1}\right|\right) + \frac{2}{n^{(\theta-1)/\theta}}, \end{aligned}$$

where $\omega_{h_n}(\delta) = \sup_{|x-y| \leqslant \delta} |h_n(x) - h_n(y)|$ is the modulus of continuity of h_n defined for all $\delta > 0$.

We justified in the proof of Corollary 15 that [Kor13, Theorem 3] applies, even if μ is not assumed to be aperiodic in our setting. This theorem tells us in particular that the rescaled height process h_n converges in distribution as n tends to infinity to some limit, in the Skhorokhod topology. Since the limit is almost surely continuous, properties of the Skhorokhod topology imply that the convergence actually holds in distribution with respect to the topology of uniform convergence. By characterization of tightness for this topology, we have for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{\delta \to 0} P\left(\omega_{h_n}(\delta) \geqslant \varepsilon\right) = 0,$$

from which we deduce that

$$D_n(x_n(U), y_n(U)) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0. \tag{38}$$

Recall that conditionally on $(T_n, \mathbf{f}_n)$, the vertices $x_n(U)$ and $y_n(U)$ have law ν_{T_n} and $\mathbf{f}_n/|\mathbf{f}_n|$ respectively. This yields using the definition (24) a bound for the Prokhorov distance between these two measures

$$d_{\mathrm{P}}^{V(T_n), D_n}\left(\nu_{T_n}, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \leqslant \inf\left\{\varepsilon > 0 \colon P\left(D_n(x_n(U), y_n(U)) \geqslant \varepsilon \;\middle|\; T_n\right) \leqslant \varepsilon\right\}.$$

In particular, we have for $\varepsilon > 0$,

$$\begin{aligned} P\left(d_{\mathrm{P}}^{V(T_n), D_n}\left(\nu_{T_n}, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \geqslant \varepsilon\right) &\leqslant P\Big(P\left(D_n(x_n(U), y_n(U)) \geqslant \varepsilon \mid T_n\right) \geqslant \varepsilon\Big) \\ &\leqslant \varepsilon^{-1} P\big(D_n(x_n(U), y_n(U)) \geqslant \varepsilon\big), \end{aligned}$$

where we used Markov's inequality to get the last upper bound. By (38), we get the convergence in probability

$$d_{\mathrm{P}}^{V(T_n), D_n}\left(\nu_{T_n}, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0.$$

By inequality (25), we deduce that

$$d_{\mathrm{GHP}}\left((V(T_n), D_n, \nu_{T_n}), \left(V(T_n), D_n, \tfrac{\mathbf{f}_n}{|\mathbf{f}_n|}\right)\right) \xrightarrow[n\to\infty,\, n\in d\mathbb{Z}]{P} 0.$$

We conclude the proof by combining the last display with Corollary 15. □

5.2.6 The spine decomposition and size-biased laws

In this section we present a *size-biasing* relation for the block-tree, in the sense of [LPP95]. Actually, we extend in a straightforward way this size-biasing relation to our setting, where we have a Galton-Watson tree and some decorations, namely the blocks. More precisely, consider the following measure on maps with a distinguished vertex of their block tree $(\mathfrak{m}, v)$

$$\mathbb{P}_u(\mathrm{d}\mathbf{M}) \sum_{v \in \mathbf{T}} \delta_v(\mathrm{d}V_\star),$$

where δ_v is the Dirac measure $A \mapsto \delta_v(A) = \mathbb{1}_{\{v \in A\}}$. Then this σ-finite measure can be decomposed as a sum of probability measures $\sum_{h \geqslant 1} \widehat{\mathbb{P}}_{u,h}(\mathrm{d}\mathbf{M}, \mathrm{d}V_\star)$, where under $\widehat{\mathbb{P}}_u^h$ the vertex $V_\star$ has height h in $\mathbf{T}$, its ancestors' degrees having *size-biased* law as defined below. The present section makes that precise.

Description of $\widehat{\mathbb{P}}_{u,h}$.

Definition 10. *Let ν be a probability distribution on $\mathbb{Z}_{\geqslant 0}$ with finite expectation m_ν. Then the* size-biased *distribution $\widehat{\nu}$ is defined by*

$$\forall k \in \mathbb{Z}_{\geqslant 0}, \quad \widehat{\nu}(k) = \frac{k\,\nu(k)}{m_\nu}.$$

When ν is a (sub-)critical offspring distribution with $\nu(0), \nu(1) \neq 0$, denote by $\left(\widehat{\mathrm{GW}}_{\nu,h}\right)_{h \geqslant 0}$ the following family of laws, on the sets of discrete trees with a distinguished vertex at height h respectively. It may be described algorithmically:

- Each vertex will either be mutant or normal, and their number of offspring are sampled independently from each other;
- Normal vertices have only normal children, whose number is sampled according to ν;
- Mutant vertices of height less than h have a number of children sampled according to the size-biased distribution $\widehat{\nu}$, all of which are normal except one, chosen uniformly, which is mutant;
- The only mutant vertex at height h reproduces like a normal vertex and is the distinguished vertex $V_\star$.

This yields a pair $(T, V_\star)$, where T is a discrete tree and $V_\star$ is a distinguished vertex of T with height h. We denote by $(V_i)_{0 \leqslant i \leqslant h_{\mathbf{T}}(V_\star)}$ the ancestor line of $V_\star$, and L_i the order of V_{i+1} in the children of V_i respectively. Observe that the construction gives that $(k_{V_i}(\mathbf{T}))_i$ are i.i.d. with law $\widehat{\nu}$, and conditionally on those variables, the variables $(L_i)_i$ are independent with uniform law on $\{1, \ldots, k_{V_i}(\mathbf{T})\}$ respectively.

We may now define the family of probability measures $(\widehat{\mathbb{P}}_{u,h})_{h \geqslant 0}$ as follows. Let $h \geqslant 0$.

- Sample $(\mathbf{T}, V_\star)$ according to the law $\widehat{\mathrm{GW}}_{\mu^u,h}$.
- For each $v \in \mathbf{T}$, sample independently and uniformly a 2-connected map $\mathfrak{b}_v^{\mathbf{M}}$ with $k_v(\mathbf{T})/2$ edges.

- Build the map $\mathbf{M}$ whose block decomposition is $(b_v^{\mathbf{M}})_{v\in\mathbf{T}}$, and $\mathbf{Q}=\varphi(\mathbf{M})$ its image by Tutte's bijection.

We are now equipped to state the size-biasing relation.

Proposition 18. *For $u\geqslant u_C$, the σ-finite measure $\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathbf{T}}\delta_v(\mathrm{d}V_\star)$ on maps with a distinguished vertex of their block-tree decomposes as the following sum of probability measures,*

$$\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathbf{T}}\delta_v(\mathrm{d}V_\star)=\sum_{h\geqslant 0}\widehat{\mathbb{P}}_{u,h}(\mathrm{d}\mathbf{M},\mathrm{d}V_\star).$$

Proof. The standard size-biasing relation for (sub-)critical Galton-Watson trees reads

$$\mathrm{GW}_\nu(\mathrm{d}\mathfrak{t})\sum_{v\in\mathfrak{t}}\delta_v(\mathrm{d}v_\star)=\sum_{h\geqslant 0}(m_\nu)^h\cdot\delta_h(h_\mathfrak{t}(v_\star))\cdot\widehat{\mathrm{GW}}_{\nu,h}(\mathrm{d}\mathfrak{t},\mathrm{d}v_\star).$$

When $u\geqslant u_C$, the offspring distribution μ^u is critical, so $m_{\mu^u}=1$. Specializing the last display to $\nu=\mu^u$ and to the value of $\mathfrak{t}$ corresponding to the block-tree of some map $\mathfrak{m}$, this gives for all such $(\mathfrak{m},v_\star)$,

$$\mathrm{GW}_{\mu^u}(\mathfrak{t})=\sum_{h\geqslant 0}\delta_h(h_\mathfrak{t}(v_\star))\widehat{\mathrm{GW}}_{\mu^u,h}(\mathfrak{t},v_\star).$$

Therefore, if we multiply both sides by $\prod_{v\in\mathfrak{t}}\frac{1}{b_{k_v(\mathfrak{t})/2}}$, we get the following by Proposition 7:

$$\mathbb{P}_u(\mathfrak{m})=\sum_{h\geqslant 0}\delta_h(h_\mathfrak{t}(v_\star))\cdot\widehat{\mathrm{GW}}_{\mu^u,h}(\mathfrak{t},v_\star)\cdot\prod_{v\in\mathfrak{t}}\frac{1}{b_{k_v(\mathfrak{t})/2}}=\sum_{h\geqslant 0}\widehat{\mathbb{P}}_{u,h}(\mathfrak{m},v_\star).$$

Since $\sum_{v\in\mathfrak{t}}\delta_v(v_\star)=1$, the last display expresses the measure $\mathbb{P}_u(\mathrm{d}\mathbf{M})\sum_{v\in\mathfrak{t}}\delta_v(\mathrm{d}V_\star)$ as a sum of the probability measures $(\widehat{\mathbb{P}}_{u,h})_{h\geqslant 0}$. □

Probabilistic properties of $\widehat{\mathbb{P}}_{u,h}$. Since we need metric information on blocks whose size follows the size-biased law $\widehat{\mu}^u$, let us introduce adequate notation. Let $u\geqslant u_C$. Denote by $\widehat{\xi}_u$ a sample of the distribution $\widehat{\mu}^u$ on some probability space (Ω,P). Then jointly define the random variables $\widehat{\mathbf{B}}_u^{\mathrm{map}}$ and $\widehat{\mathbf{B}}_u^{\mathrm{quad}}$ as sampled uniformly among respective blocks with size $\widehat{\xi}_u/2$, in such a way that they are linked by Tutte's bijection, i.e. their joint law satisfies

$$\left(\widehat{\mathbf{B}}_u^{\mathrm{map}},\widehat{\mathbf{B}}_u^{\mathrm{quad}}\right)\overset{(d)}{=}\left(B^{\mathrm{map}}_{\widehat{\xi}_u/2},B^{\mathrm{quad}}_{\widehat{\xi}_u/2}\right).$$

Furthermore, conditionally on $\widehat{\xi}_u$, sample independently U a uniform label in $\{1,\dots,\widehat{\xi}_u\}$. This yields the following 4-tuple

$$\left(\widehat{\xi}_u\,,\,\widehat{\mathbf{B}}_u^{\mathrm{map}}\,,\,\widehat{\mathbf{B}}_u^{\mathrm{quad}}\,,\,U\right).$$

Lemma 19. *For all $h\geqslant 1$, we have the identity in law*

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_i}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_i}\,,\,L_i\right)_{0\leqslant i<h}\,;\,\widehat{\mathbb{P}}_{u,h}\right)=\left[\mathrm{Law}\left(\left(\widehat{\xi}_u\,,\,\widehat{\mathbf{B}}_u^{\mathrm{map}}\,,\,\widehat{\mathbf{B}}_u^{\mathrm{quad}}\,,\,U\right)\,;\,P\right)\right]^{\otimes h}$$

where $\mathrm{Law}(X;Q)$ *is the law of* X *under* Q.

Proof. Recall that under $\widehat{\mathbb{P}}_{u,h}$, the pair $(\mathbf{T}, V_\star)$ has law $\widehat{\mathrm{GW}}_{\mu^u,h}$. By definition of the law $\widehat{\mathrm{GW}}_{\mu^u,h}$, the ancestor line of the distinguished vertex $V_\star$ in $\mathbf{T}$ is made of mutant vertices. This means that the family $(k_{v_i}(\mathbf{T}))_{0,\leqslant i<h}$ is i.i.d. sampled from the size-biased distribution $\widehat{\mu}^u$, which is the law of $\widehat{\xi}_u$, and that independently of each other, each V_{i+1} has uniform rank L_i among the $k_{V_i}(\mathbf{T})$ children of V_i. Hence we have the identity in law

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T})\ ,\ L_i\right)_{0\leqslant i<h}\ ;\ \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\left((\widehat{\xi}_u\ ,\ U);P\right)\right]^{\otimes h}.$$

Now under $\widehat{\mathbb{P}}_{u,h}$ the conditional law of the blocks $(\mathfrak{b}_v^{\mathbf{M}})_{v\in\mathbf{T}}$ with respect to $\mathbf{T}$ is that of independent blocks, sampled uniformly from blocks with size $(k_V(\mathbf{T})/2)_{v\in\mathbf{T}}$ respectively. In particular, the blocks $(\mathfrak{b}_{V_i}^{\mathbf{M}})_{0\leqslant i<h}$ are sampled independently, uniformly from blocks with size $(k_{V_i}(\mathbf{T})/2)_{0\leqslant i<h}$ respectively. Therefore the preceding identity in law extends to the following one

$$\mathrm{Law}\left(\left(k_{V_i}(\mathbf{T})\ ,\ \mathfrak{b}_{V_i}^{\mathbf{M}}\ ,\ L_i\right)_{0\leqslant i<h}\ ;\ \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\left((\widehat{\xi}_u\ ,\ \widehat{\mathbf{B}}_u^{\mathrm{map}}\ ,\ U);P\right)\right]^{\otimes h}.$$

Finally, recall from Proposition 5 that $\mathfrak{b}_{V_i}^{\mathbf{Q}}$ is the image of $\mathfrak{b}_{V_i}^{\mathbf{M}}$ by Tutte's bijection. Since by definition $\widehat{\mathbf{B}}_u^{\mathrm{quad}}$ is also the image of $\widehat{\mathbf{B}}_u^{\mathrm{map}}$ by this bijection, the identity in law extends to the one in the proposition. □

We get in particular from Lemma 19 that the variables $\left(D(\mathfrak{b}_{V_i}^{\mathbf{M}}, L_i)\right)_{0\leqslant i<h}$ are i.i.d. under $\widehat{\mathbb{P}}_{u,h}$. It is a bit less clear that the variables $\left(D_{\mathbf{Q}}(\mathfrak{b}_{V_i}^{\mathbf{Q}}, L_i)\right)_{0\leqslant i<h}$ from Lemma 13 are also i.i.d., since they seem to simultaneously depend on global metric properties of $\mathbf{Q}$.

Lemma 20. *Denote by* $D(\mathfrak{b}, l)$ *the distance in a simple quadrangulation* $\mathfrak{b}$ *between its root vertex and the closest endpoint of the* l*-th edge in the order induced by the block-tree decomposition, the same order as the one introduced before Lemma 13. Then for all* $h \geqslant 1$*, there is the identity in law*

$$\mathrm{Law}\left((D_{\mathbf{Q}}(\mathfrak{b}_{V_i}^{\mathbf{Q}}, L_i))_{0\leqslant i<h}\ ;\ \widehat{\mathbb{P}}_{u,h}\right) = \left[\mathrm{Law}\Big(D(\widehat{\mathbf{B}}_u^{\mathrm{quad}}, U)\ ;\ P\Big)\right]^{\otimes h}.$$

Proof. Recall from the notation introduced for Lemma 13 that for $\mathfrak{b}$ a simple block of a quadrangulation $\mathfrak{q}$, and l an integer in $\{1,\dots,2|\mathfrak{b}|\}$, $D_{\mathfrak{q}}(\mathfrak{b}, l)$ is the graph distance in $\mathfrak{b}$ between the endpoints of the l-th edge of $\mathfrak{b}$ in breadth-first order, and the endpoint of the root edge of $\mathfrak{b}$ which is closest to the root vertex of $\mathfrak{q}$.

Denote by $\mathfrak{b} \mapsto F(\mathfrak{b})$ the mapping which reverses the rooted oriented edge of a simple quadrangulation. Introduce also for $\mathfrak{b}$ a simple quadrangulation, $f_{\mathfrak{b}}$ the permutation of $\{1,\dots,2|\mathfrak{b}|\}$ which maps the breadth-first order on $\mathfrak{b}$ to the breadth-first-order on $F(\mathfrak{b})$. Finally, define the event $\mathcal{E}_i$ that the root vertex of $b_{V_i}^{\mathbf{Q}}$ is closer to the root vertex of $\mathbf{Q}$ than the other endpoint of the root edge of $b_{V_i}^{\mathbf{Q}}$. Then by definition, for all $0 \leqslant i < h$ we have that

$$D_{\mathbf{Q}}(\mathfrak{b}_{V_i}^{\mathbf{Q}}, L_i) = \mathbb{1}_{\mathcal{E}_i} \cdot D\Big(\mathfrak{b}_{V_i}^{\mathbf{Q}}, L_i\Big) + (1 - \mathbb{1}_{\mathcal{E}_i}) \cdot D\Big(F(\mathfrak{b}_{V_i}^{\mathbf{Q}}), f_{\mathfrak{b}_{V_i}^{\mathbf{Q}}}(L_i)\Big).$$

Let $\mathcal{F}_i$ denote the sigma-algebra of the variables $(k_{V_j}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_j}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_j}\,,\,L_j)_{0\leqslant j<i}$. Then by Lemma 19, we have that the tuple $(k_{V_i}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_i}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_i}\,,\,L_i)$ is independent of $\mathcal{F}_i$, and has the same law as $\big(\widehat{\xi}_u\,,\,\widehat{\mathbf{B}}^{\mathrm{map}}_u\,,\,\widehat{\mathbf{B}}^{\mathrm{quad}}_u\,,\,U\big)$. Now the crucial point is that the event $\mathcal{E}_i$ is $\mathcal{F}_i$-measurable, since it can be decided whether or not it holds by looking only at the first i blocks on the spine. In particular it is independent of $(k_{V_j}(\mathbf{T})\,,\,\mathfrak{b}^{\mathbf{M}}_{V_j}\,,\,\mathfrak{b}^{\mathbf{Q}}_{V_j}\,,\,L_j)_{j\geqslant i}$. This implies the following

$$\begin{aligned}&\mathrm{Law}\left(\Big(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_i},L_i)\Big)_{0\leqslant i<h}\,;\,\widehat{\mathbb{P}}_{u,h}\right)\\&=\bigotimes_{0\leqslant i<h}\left[\widehat{\mathbb{P}}_{u,h}(\mathcal{E}_i)\cdot\mathrm{Law}\Big(D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u,U)\Big)+(1-\widehat{\mathbb{P}}_{u,h}(\mathcal{E}_i))\cdot\mathrm{Law}\Big(D\big(F(\widehat{\mathbf{B}}^{\mathrm{quad}}_u),f_{\widehat{\mathbf{B}}^{\mathrm{quad}}_u}(U)\big)\Big)\right].\end{aligned}$$

The proposition is therefore proved if we justify the identity in law

$$D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u,U)\overset{(d)}{=}D\big(F(\widehat{\mathbf{B}}^{\mathrm{quad}}_u),f_{\widehat{\mathbf{B}}^{\mathrm{quad}}_u}(U)\big).\tag{39}$$

To check this, first notice that F is a bijection since it is involutive, so that in particular the uniform law on simple quadrangulations with k edges is invariant under F. By definition, for $\mathfrak{b}$ a simple quadrangulation, $f_{\mathfrak{b}}$ is also a bijection so that the uniform measure on $\{1,\dots,2|\mathfrak{b}|\}$ is invariant under it. Denoting U_k a uniform random variable on $\{1,\dots,2k\}$, this gives for each $k\geqslant 1$ the identity in law

$$D(B^{\mathrm{quad}}_k,U_k)\overset{(d)}{=}D\big(F(B^{\mathrm{quad}}_k),f_{B^{\mathrm{quad}}_k}(U_k)\big).$$

Since the pair $(\widehat{\mathbf{B}}^{\mathrm{quad}}_u,U)$ is the $\widehat{\xi}_u/2$-mixture of the laws $(B_k,U_k)_{k\geqslant 1}$, the identity in law (39) also holds and this conludes the proof. □

Moments of typical distances in a size-biased block. We may now examine how fat are the tails of this i.i.d. family of distances along the spine, which we wish to sum.

Proposition 21. *Let D be either the variable $D(\widehat{\mathbf{B}}^{\mathrm{map}}_u,U)$ or $D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u,U)$. Then for $u>u_C$, there exists $\varepsilon>0$ such that $E[\exp(tD)]<\infty$ for all real $t<\varepsilon$. And for $u=u_C$, we have $E\left[D^\beta\right]<\infty$ for all $0<\beta<2$.*

Proof. The variable D is defined as a distance in $\widehat{\mathbf{B}}_u$, where $\widehat{\mathbf{B}}_u$ is either $\widehat{\mathbf{B}}^{\mathrm{map}}_u$ or $\widehat{\mathbf{B}}^{\mathrm{quad}}_u$. Hence it suffices to prove that the above moments are finite when we replace D by $\mathrm{diam}(\widehat{\mathbf{B}}_u)$.

Let $u>u_C$. Then $\mathrm{diam}(\widehat{\mathbf{B}}_u)\leqslant\widehat{\xi}_u$, and the latter variable has finite exponential moments since $P(\widehat{\xi}_u\geqslant x)=\sum_{2j\geqslant x}2j\mu^u(\{2j\})$, where μ^u has a tail decaying exponentially fast by Theorem 1.

Now take $u=u_C=9/5$ and let $\delta\in(0,2)$. Also let $\varepsilon>0$ to be chosen later depending on δ. Using the notation B_k for B^{map}_k or B^{quad}_k, set

$$p_\varepsilon(k)=P\left(\mathrm{diam}(B_k)\geqslant k^{1/4+\varepsilon}\right).$$

By Proposition 10, we have that $p_\varepsilon(k)$ decays stretched exponentially as $k\to\infty$. Therefore we get a constant $C>0$ such that $k^2p_\varepsilon(k)\leqslant C$ for all k. Recall that we have $\mathrm{diam}(\widehat{\mathbf{B}}_u)\leqslant\widehat{\xi}_u$. Distinguishing

upon whether $\mathrm{diam}(\widehat{\mathbf{B}}_u) \leqslant (\widehat{\xi}_u)^{1/4+\varepsilon}$ or $\mathrm{diam}(\widehat{\mathbf{B}}_u) > (\widehat{\xi}_u)^{1/4+\varepsilon}$ and taking a conditional expectation with respect to $\widehat{\xi}_u$, we get

$$\begin{aligned} E\left[\left(\mathrm{diam}(\widehat{\mathbf{B}}_u)\right)^{2-\delta}\right] &\leqslant E\left[\left((\widehat{\xi}_u)^{1/4+\varepsilon}\right)^{2-\delta} \mathbb{1}_{\{\mathrm{diam}(\widehat{\mathbf{B}}_u)\leqslant(\widehat{\xi}_u)^{1/4+\varepsilon}\}}\right] + E\left[(\widehat{\xi}_u)^{2-\delta}\, p_\varepsilon\Big(\widehat{\xi}_u\Big)\right] \\ &\leqslant E\left[(\widehat{\xi}_u)^{(1/4+\varepsilon)(2-\delta)}\right] + C \\ &= \sum_{2j\geqslant 0} (2j)^{(1/4+\varepsilon)(2-\delta)} \cdot 2j\mu^{u_C}(\{2j\}) + C. \end{aligned}$$

If ε is small enough so that $(1/4+\varepsilon)(2-\delta) < 1/2$, then the last sum is finite since by Theorem 1 we have $\mu^{u_C}(\{2j\}) = O(j^{-5/2})$. Therefore $E\left[\left(\mathrm{diam}(\widehat{\mathbf{B}}_u)\right)^{2-\delta}\right] < \infty$. □

Let us make a brief commentary, and justify that when $u = u_C$, Proposition 21 is optimal, in the sense that $D(\widehat{\mathbf{B}}^{\mathrm{quad}}_{u_C}, U)$ does not have moments of order β for $\beta \geqslant 2$. Firstly, one easily checks that functionals on pointed measured metric spaces of the form

$$(X, x_0, d_X, \nu_X) \mapsto \int_X \nu_X(\mathrm{d}x)\left(d_X(x_0,x)\right)^\beta$$

are continuous with respect to the Gromov-Hausdorff-Prokhorov topology. Addario-Berry and Albenque [ABA21] prove the GHP convergence of size k uniform simple quadrangulations, rescaled by $\mathrm{cst}\cdot k^{-1/4}$, to the measured Brownian sphere $(\mathcal{S}, D^*, \lambda)$. This holds when putting either the uniform measure on vertices of B^{quad}_k or the size-biased one by [ABW17]. In particular, by the abovementioned continuity, we have the convergence in distribution

$$E\left[\left(\mathrm{cst}\cdot k^{-1/4} D(B^{\mathrm{quad}}_k, U_k)\right)^\beta \;\middle|\; B^{\mathrm{quad}}_k\right] \xrightarrow[k\to\infty]{(d)} \int_{\mathcal{S}} \lambda(\mathrm{d}x)\big(D^*(x_0,x)\big)^\beta,$$

where U_k is uniform on $\{1,\dots,2k\}$ and x_0 is the distinguished point on the Brownian sphere. Since the variable $\int_{\mathcal{S}} \lambda(\mathrm{d}x)\big(D^*(x_0,x)\big)^\beta$ is almost surely positive, the left-hand-side forms a tight sequence of $(0,\infty)$-valued random variables. Therefore it is bounded away from 0 with *uniform* positive probability. This implies a lower bound $E\left[D(B^{\mathrm{quad}}_k, U_k)^\beta\right] \geqslant c(k^{1/4})^\beta$, for some $c = c(\beta) > 0$. In particular,

$$\begin{aligned} E\left[D(\widehat{\mathbf{B}}^{\mathrm{quad}}_{u_C}, U)^\beta\right] = \sum_{2j\geqslant 0} E\left[D(B^{\mathrm{quad}}_j, U_j)^\beta\right]\cdot \widehat{\mu}^{u_C}(\{2j\}) &\geqslant \sum_{2j\geqslant 0} cj^{\beta/4}\cdot 2j\mu^{u_C}(\{2j\}) \\ &= \sum_{2j\geqslant 0} \Theta(j^{\beta/4+1-5/2}). \end{aligned}$$

The latter sum is infinite when $\beta \geqslant 2$, which proves that $D(\widehat{\mathbf{B}}^{\mathrm{quad}}_{u_C}, U)$ does not have moments of order β for $\beta \geqslant 2$. The same argument would hold for $D(\widehat{\mathbf{B}}^{\mathrm{map}}_{u_C}, U)$, but we lack at the moment the GHP convergence of size-k uniform 2-connected maps.

5.2.7 Moderate deviations estimate

When increments of a random walk possess only a polynomial moment of order $\beta > 1$, as is the case of $D(\widehat{\mathbf{B}}^{\mathrm{map}}_u, U)$ and $D(\widehat{\mathbf{B}}^{\mathrm{quad}}_u, U)$ when $u = u_C$, moderate and large deviation events can possibly

have probabilities which decay slowly, that is polynomially with n. In the case of heavy-tailed increments, this indeed happens since those moderate and large deviation events can be realised by taking one large increment. This *one-big-jump* behaviour is actually precisely how these large deviations events are realised. This phenomenon, which we have already encountered in Section 3 for $u < u_C$, is known as condensation. For a more precise statement, see [Jan12, AL09, AB19].

One could hope that if we prevent the variables from condensating, we could still get stretched-exponentially small probabilities for large deviation events. We make this precise in the following proposition, by stating that this is the case when we suitably truncate the increments. We were not able to find an instance of such an estimate in the literature, although it has certainly been encountered in some form. We thus include a short proof, which as usual relies on a Chernoff bound.

Proposition 22. *Let X be a real random variable with i.i.d. copies $(X_i)_{i\geqslant 1}$. Assume that there exists $\beta \in (1,2]$ such that $E\left[|X|^\beta\right] < \infty$ and that we have $E[X] = 0$.*

Then, for all $\delta > 0$, $\gamma \in (0, 1/\beta + \delta)$, and $\nu \in \big(0, \delta \wedge (1/\beta + \delta - \gamma)\big)$, there exists a constant $C > 0$ such that for all $n \geqslant 1$,

$$P\left(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} X_i \mathbb{1}_{\{X_i \leqslant n^\gamma\}} > n^{1/\beta+\delta}\right) \leqslant C\exp(-n^\nu).$$

Remark 7. *A straightforward adaptation of the proof shows that the conclusion still holds if the only assumptions on the variables $(X_i)_i$ are $E\big[X_i \;\big|\; X_1, \dots, X_{i-1}\big] \leqslant 0$ and $\sup_{i\geqslant 1} E\big[|X_i|^\beta \;\big|\; X_1, \dots, X_{i-1}\big] < \infty$.*

Proof of Proposition 22. Fix an arbitrary θ such that $\max(\gamma, 1/\beta) < \theta < 1/\beta + \delta$. By Chernoff's bound, we get for all $1 \leqslant k \leqslant n$,

$$\begin{aligned} P\left(\sum_{i=1}^{k} X_i \mathbb{1}_{\{X_i\leqslant n^\gamma\}} > n^{1/\beta+\delta}\right) &\leqslant \exp(-n^{1/\beta+\delta-\theta})\Big(E\left[\exp\big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\big)\right]\Big)^k \\ &\leqslant \exp(-n^{1/\beta+\delta-\theta})\Big(1 \vee E\left[\exp\big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\big)\right]\Big)^n. \end{aligned}$$

Therefore we obtain by a union bound the estimate

$$P\left(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} X_i \mathbb{1}_{\{X_i\leqslant n^\gamma\}} > n^{1/\beta+\delta}\right) \leqslant n\cdot\exp(-n^{1/\beta+\delta-\theta})\Big(1 \vee E\left[\exp\big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\big)\right]\Big)^n.$$

Since θ is arbitrary in the interval $\big(\max(\gamma, 1/\beta)\,,\, 1/\beta+\delta\big)$, the exponent $\nu := 1/\beta+\delta-\theta$ is arbitrary in the interval $\big(0, \delta \wedge (1/\beta + \delta - \gamma)\big)$. As a consequence, to prove the proposition it is sufficient to show that

$$E\left[\exp\Big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\Big)\right] \leqslant 1 + O(n^{-1}). \tag{40}$$

Notice that since $\beta \in (1,2]$, for all $M > 0$ the following inequality holds for t near 0 or $-\infty$:

$$\exp(t) \leqslant 1 + t + M|t|^\beta.$$

Therefore, if one takes M large enough it holds for all $t \in (-\infty, 1]$. Fix such a constant M.

Given $\lambda, s \geqslant 0$, distinguishing upon whether $\lambda x \in (-\infty, 1]$ or not and using that $x\mathbb{1}_{\{x\leqslant s\}} \leqslant x$ and $x\mathbb{1}_{\{\lambda x\leqslant 1\}} \leqslant x$ (even when $x < 0$), we get for all $x \in \mathbb{R}$,

$$\begin{aligned}\exp\Big(\lambda x\mathbb{1}_{\{x\leqslant s\}}\Big) &\leqslant \Big(1 + \lambda x\mathbb{1}_{\{x\leqslant s\}} + M\lambda^\beta |x|^\beta \mathbb{1}_{\{x\leqslant s\}}\Big)\cdot \mathbb{1}_{\{\lambda x\leqslant 1\}} + \exp(\lambda x\mathbb{1}_{\{x\leqslant s\}})\cdot \mathbb{1}_{\{\lambda x>1\}} \\ &\leqslant 1 + \lambda x + M\lambda^\beta |x|^\beta + \exp(\lambda s)\cdot \mathbb{1}_{\{\lambda x>1\}}. \end{aligned} \tag{41}$$

Applying this inequality with $x = X$, $\lambda = n^{-\theta}$, $s = n^\gamma$ and taking expectations we obtain

$$E\left[\exp\Big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\Big)\right] \leqslant 1 + n^{-\theta}E\left[X\right] + Mn^{-\beta\theta}E\left[|X|^\beta\right] + \exp\big(n^{\gamma-\theta}\big)\cdot P\left(X \geqslant n^\theta\right).$$

Recall that $E[X] = 0$ by hypothesis, that $\gamma - \theta < 0$ by choice of θ, and use Markov's inequality. This yields

$$E\left[\exp\Big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\Big)\right] \leqslant 1 + 0 + Mn^{-\beta\theta}E\left[|X|^\beta\right] + \exp(1)\cdot n^{-\beta\theta}E\left[|X|^\beta\right].$$

Since by hypothesis $E\left[|X|^\beta\right] < \infty$, we have

$$E\left[\exp\Big(n^{-\theta}X\mathbb{1}_{\{X\leqslant n^\gamma\}}\Big)\right] \leqslant 1 + O(n^{-\beta\theta}) \leqslant 1 + O(n^{-1})$$

where the last inequality comes from the choice of θ, which is greater than $1/\beta$. Therefore (40) is satisfied and the proposition is proved. □

5.2.8 A lemma to compare $\mathfrak{m}$, $\mathfrak{q}$ and $\mathfrak{t}$

Let us state a lemma that elaborates on the additivity of distances on consecutive blocks, so that we can bound the GHP-distance between a map (resp. a quadrangulation) and its block-tree scaled by some constant. Let κ_1 and κ_2 be positive constants. Let $\mathfrak{m}$ be a map, $\mathfrak{q}$ its associated quadrangulation by Tutte's bijection, and $\mathfrak{t}$ their block-tree.

For x a vertex of either $\mathfrak{m}$ or $\mathfrak{q}$, denote as in Lemmas 12 and 13 by $v_\star$ the vertex v of $\mathfrak{t}$ closest to the root of $\mathfrak{t}$ such that x is a vertex of $\mathfrak{b}_v^\mathfrak{m}$ (resp. $\mathfrak{b}_v^\mathfrak{q}$). Set similarly $h_\star = h_\mathfrak{t}(v_\star)$ the height of $v_\star$ in $\mathfrak{t}$, and $(v_i)_{0\leqslant i\leqslant h_\star}$ the ancestor line of $v_\star$ in $\mathfrak{t}$, with v_0 the root of $\mathfrak{t}$ and $v_{h_\star} = v_\star$. Also denote by x_i the root vertex of $\mathfrak{b}_{v_i}^\mathfrak{m}$ (resp. $\mathfrak{b}_{v_i}^\mathfrak{q}$). Finally, let $(l_i)_{0\leqslant i<h_\star}$ be the respective breadth-first index of the corner in $\mathfrak{b}_{v_i}^\mathfrak{m}$ (resp. the edge in $\mathfrak{b}_{v_i}^\mathfrak{q}$) to which the root corner of $\mathfrak{b}_{v_{i+1}}^\mathfrak{m}$ (resp. the root edge of $\mathfrak{b}_{v_{i+1}}^\mathfrak{q}$) is attached. Finally, denote by $\Delta(\mathfrak{m})$ (resp. $\Delta(\mathfrak{q})$) the largest diameter of a block of $\mathfrak{m}$ (resp. $\mathfrak{q}$). We set the quantities

$$R(\mathfrak{m}, v_\star, \kappa_1) = \max_{0\leqslant i<h_\star}\left|\sum_{j=i}^{h_\star-1}\Big(D(\mathfrak{b}_{v_j}^\mathfrak{m}, l_j) - \kappa_1\Big)\right|,$$

$$\text{and}\quad R(\mathfrak{q}, v_\star, \kappa_2) = \max_{0\leqslant i<h_\star}\left|\sum_{j=i}^{h_\star-1}\Big(D_\mathfrak{q}(\mathfrak{b}_{v_j}^\mathfrak{q}, l_j) - \kappa_2\Big)\right|.$$

Notice that the preceding quantities depend on x only through $v_\star$ and therefore make sense as functions of only $(\mathfrak{m}, v_\star, \kappa_1)$ and $(\mathfrak{q}, v_\star, \kappa_2)$ respectively.

Lemma 23. *Let $\mathbf{f}^{\mathfrak{m}}$ and $\mathbf{f}^{\mathfrak{q}}$ be the functions on $V(\mathfrak{t})$ defined by $\mathbf{f}^{\mathfrak{m}}(v) = |V(\mathfrak{b}_v^{\mathfrak{m}})| - 1$ and $\mathbf{f}^{\mathfrak{q}}(v) = |V(\mathfrak{b}_v^{\mathfrak{q}})| - 2$ respectively. With the above notation, we have for all $\varepsilon > 0$,*

$$\mathrm{d_{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{m}}\,,\, \varepsilon\kappa_1 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{m}}}{|\mathbf{f}^{\mathfrak{m}}|}\right)\right) \leqslant \kappa_1\varepsilon + \frac{3\varepsilon}{2}\Delta(\mathfrak{m}) + \varepsilon \max_{v_\star \in V(\mathfrak{t})} R(\mathfrak{m}, v_\star, \kappa_1) + \frac{2}{|\mathbf{f}^{\mathfrak{m}}|},$$

and

$$\mathrm{d_{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{q}}\,,\, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right)\right) \leqslant (\kappa_2 + 3)\varepsilon + \frac{3\varepsilon}{2}\Delta(\mathfrak{q}) + \varepsilon \max_{v_\star \in V(\mathfrak{t})} R(\mathfrak{q}, v_\star, \kappa_2) + \frac{4}{|\mathbf{f}^{\mathfrak{q}}|}.$$

Proof. Let us first treat the inequality involving $\mathfrak{q}$, which is a bit more involved. Consider the correspondence C between $V(\mathfrak{q})$ and $V(\mathfrak{t})$ defined as follows. A vertex x of $\mathfrak{q}$ is set in correspondence with a vertex v of $\mathfrak{t}$ if and only if v is the vertex $v_\star$ defined as above from x; and let $\rho(x) := v_\star$. Put differently, a vertex v of $\mathfrak{t}$ is put in correspondence precisely with the $\mathbf{f}^{\mathfrak{q}}(v) = |V(\mathfrak{b}_v^{\mathfrak{q}})| - 2$ vertices of the block $\mathfrak{b}_v^{\mathfrak{q}}$ which are not incident to the root edge of $\mathfrak{b}_v^{\mathfrak{q}}$ (except when v is the root vertex in which case v is in correspondence with all the vertices of $\mathfrak{b}_v^{\mathfrak{q}}$).

Let γ be the uniform measure on the previously defined set $C = \{(x, \rho(x))\colon x \in V(\mathfrak{q})\} \subset V(\mathfrak{q}) \times V(\mathfrak{t})$. Let the function $\pi\colon C \to V(\mathfrak{t})$ be the restriction of the projection $\pi_{\mathfrak{t}} : (x, y) \in V(\mathfrak{q}) \times V(\mathfrak{t}) \mapsto y \in V(\mathfrak{t})$. The preimages of π have cardinal $|\pi^{-1}(v)| = |\rho^{-1}(v)| = \mathbf{f}^{\mathfrak{q}}(v) + 2\delta_{\mathrm{root}}(v)$, where $\delta_{\mathrm{root}}(v)$ is an indicator that x is incident to the root-edge of $\mathfrak{q}$. Tautologically, the measure γ defines a coupling between its images by the projections $\pi_{\mathfrak{q}} : (x, y) \in V(\mathfrak{q}) \times V(\mathfrak{t}) \mapsto x \in V(\mathfrak{q})$ and $\pi_{\mathfrak{t}}$. That is to say, γ is a coupling of the measures $\nu_{\mathfrak{q}}$ and $\frac{\mathbf{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}|+2}$. It is also supported by C, *i.e.* $\gamma\big((V(\mathfrak{q}) \times V(\mathfrak{t})) \setminus C\big) = 0$. By the triangle inequality and the preceding observations, we have

$$\begin{aligned}
&d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{q}}\,,\, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right)\right)\\
&\leqslant d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{q}}\,,\, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}|+2}\right)\right) + d_{\mathrm{GHP}}\left(\varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}|+2}\right),\, \varepsilon\kappa_2 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right)\right)\\
&\leqslant \frac{\varepsilon}{2}\,\mathrm{dis}(C; d_{\mathfrak{q}}, \kappa_2 d_{\mathfrak{t}}) + d_{\mathrm{P}}^{(V(\mathfrak{t}), \varepsilon\kappa_2 d_{\mathfrak{t}})}\left(\tfrac{\mathbf{f}^{\mathfrak{q}}+2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}|+2}, \tfrac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right).
\end{aligned}$$

The last inequality uses (25) to bound the second GHP distance by a Prokhorov distance. Now, the Prokhorov distance between two measures is bounded by their total variation distance, and for measures μ and ν we have elementarily $d_{\mathrm{TV}}(\frac{\mu+\nu}{|\mu|+|\nu|}, \frac{\mu}{|\mu|}) \leqslant \frac{2|\nu|}{|\mu|}$. Therefore, we have

$$d_{\mathrm{P}}^{(V(\mathfrak{t}), \varepsilon\kappa_2 d_{\mathfrak{t}})}\left(\frac{\mathbf{f}^{\mathfrak{q}} + 2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}| + 2}, \frac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right) \leqslant d_{\mathrm{TV}}\left(\frac{\mathbf{f}^{\mathfrak{q}} + 2\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{q}}| + 2}, \frac{\mathbf{f}^{\mathfrak{q}}}{|\mathbf{f}^{\mathfrak{q}}|}\right) \leqslant \frac{4}{|\mathbf{f}^{\mathfrak{q}}|}.$$

It remains to bound the distortion $\mathrm{dis}(C; d_{\mathfrak{q}}, \kappa_2 d_{\mathfrak{t}})$. This amounts to bounding $|d_{\mathfrak{q}}(x, \widetilde{x}) - \kappa_2 d_{\mathfrak{t}}(\rho(x), \rho(\widetilde{x}))|$ uniformly for all pairs of vertices $(x, \widetilde{x})$ in $V(\mathfrak{q}) \times V(\mathfrak{q})$. Let x and $\widetilde{x}$ be vertices of $\mathfrak{q}$. As before, we define the vertices $v_\star = \rho(x)$ and $\widetilde{v}_\star = \rho(\widetilde{x})$ in $V(\mathfrak{t})$, their respective heights $h_\star$ and $\widetilde{h}_\star$, their ancestor lines $(v_i)_{0 \leqslant i \leqslant h_\star}$ and $(\widetilde{v}_i)_{0 \leqslant i \leqslant h_\star}$, the labels $(l_i)_{0 \leqslant i < h_\star}$ and $(\widetilde{l}_i)_{0 \leqslant i < h_\star}$, and the vertices $(x_i)_{0 \leqslant i \leqslant h_\star}$ and $(\widetilde{x}_i)_{0 \leqslant i \leqslant h_\star}$.

Let i be such that $v_i = \widetilde{v}_i$ is the last common ancestor of $v_\star$ and $\widetilde{v}_\star$ in $\mathfrak{t}$. First notice that there exists $\delta_0 \in \{0, \pm 1, \pm 2\}$ such that

$$d_{\mathfrak{q}}(x, \widetilde{x}) = \delta_0 + d_{\mathfrak{q}}(x, x_{i+1}) + d_{\mathfrak{b}_{v_i}^{\mathfrak{q}}}(x_{i+1}, \widetilde{x}_{i+1}) + d_{\mathfrak{q}}(\widetilde{x}_{i+1}, \widetilde{x}). \tag{42}$$

Indeed, similarly as in the proof of Lemma 13, a geodesic from x to $\widetilde{x}$ must visit, once and in that order,

- the vertex x,
- either x_{i+1}, or x'_{i+1} the other endpoint of the root-edge of $\mathfrak{b}^{\mathfrak{q}}_{v_{i+1}}$,
- either $\widetilde{x}_{i+1}$, or $\widetilde{x}'_{i+1}$ the other endpoint of the root-edge of $\mathfrak{b}^{\mathfrak{q}}_{\widetilde{v}_{i+1}}$,
- the vertex $\widetilde{x}$.

Since x_{i+1} and x'_{i+1}, as well as $\widetilde{x}_{i+1}$ and $\widetilde{x}'_{i+1}$, are at distance 1 respectively, and since a geodesic between points in $\mathfrak{b}_{v_i}$ must stay in $\mathfrak{b}_{v_i}$, we get that (42) holds, for some $\delta_0 \in \{0, \pm 1, \pm 2\}$. Then, Lemma 13 allows to decompose the distances $d_{\mathfrak{q}}(x, x_{i+1})$ and $d_{\mathfrak{q}}(\widetilde{x}_{i+1}, \widetilde{x})$, with some $\delta, \widetilde{\delta}$ in $\{0, 1\}$. Combining this with (42), this gives

$$\begin{aligned} d_{\mathfrak{q}}(x, \widetilde{x}) &= \delta_0 + \left[\delta + d_{\mathfrak{b}^{\mathfrak{q}}_{v_\star}}(x, x_{h_\star}) + \sum_{i+1 \leqslant j < h_\star - 1} D_{\mathfrak{q}}(v_j, l_j)\right] + d_{b^{\mathfrak{q}}_{v_i}}(x_{i+1}, \widetilde{x}_{i+1}) \\ &\quad + \left[\widetilde{\delta} + d_{\mathfrak{b}^{\mathfrak{q}}_{\widetilde{v}_\star}}(\widetilde{x}, \widetilde{x}_{\widetilde{h}_\star}) + \sum_{i+1 \leqslant j < \widetilde{h}_\star - 1} D_{\mathfrak{q}}(\widetilde{v}_j, \widetilde{l}_j)\right] \\ &= \delta_0 + \delta + \widetilde{\delta} + d_{\mathfrak{b}^{\mathfrak{q}}_{v_\star}}(x, x_{h_\star}) + d_{\mathfrak{b}^{\mathfrak{q}}_{v_i}}(x_{i+1}, \widetilde{x}_{i+1}) + d_{\mathfrak{b}^{\mathfrak{q}}_{\widetilde{v}_\star}}(\widetilde{x}, \widetilde{x}_{\widetilde{h}_\star}) \\ &\quad + \sum_{i+1 \leqslant j < h_\star - 1} (D_{\mathfrak{q}}(v_j, l_j) - \kappa_2) + \sum_{i+1 \leqslant j < \widetilde{h}_\star - 1} \big(D_{\mathfrak{q}}(\widetilde{v}_j, \widetilde{l}_j) - \kappa_2\big) \\ &\quad + \kappa_2(h_\star - i - 1) + \kappa_2(\widetilde{h}_\star - i - 1). \end{aligned}$$

The sum of the first six terms has absolute value at most $6 + 3\Delta(\mathfrak{q})$, the two sums have absolute value at most $R(\mathfrak{q}, x, \kappa_2)$ and $R(\mathfrak{q}, \widetilde{x}, \kappa_2)$ respectively, and the two remaining terms sum to $\kappa_2(h_\star - i - 1) + \kappa_2(\widetilde{h}_\star - i - 1) = \kappa_2 d_{\mathfrak{t}}(v, \widetilde{v}) - 2\kappa_2$. Therefore by the triangle inequality,

$$|d_{\mathfrak{q}}(x, \widetilde{x}) - \kappa_2 d_{\mathfrak{t}}(v, \widetilde{v})| \leqslant 2\kappa_2 + 6 + 3\Delta(\mathfrak{q}) + R(\mathfrak{q}, x, \kappa_2) + R(\mathfrak{q}, \widetilde{x}, \kappa_2).$$

Since this holds for every $(x, v) = (x, \rho(x)) \in C$ and $(\widetilde{x}, \widetilde{v}) = (\widetilde{x}, \rho(\widetilde{x})) \in C$, the max of the right-hand side over $x, \widetilde{x} \in V(\mathfrak{q})$ is actually a bound on the distortion $\mathrm{dis}(C; d_{\mathfrak{q}}, \kappa_2 d_{\mathfrak{t}})$, which is precisely what we needed to conclude.

For the inequality involving $\mathfrak{m}$, the reasoning is quite similar. Take C the correspondence such that $x \in V(\mathfrak{m})$ is in correspondence with $v \in V(\mathfrak{t})$ if and only if v is the vertex $\rho(x) = v_\star$. Equivalently, a vertex v of $\mathfrak{t}$ is put in correspondence with the $\mathbf{f}^{\mathfrak{m}}(v) = |V(\mathfrak{b}^{\mathfrak{m}}_v)| - 1$ non-root vertices of the block $\mathfrak{b}^{\mathfrak{m}}_v$, except when v is the root vertex of $\mathfrak{t}$ in which case v is in correspondence with all the vertices of $\mathfrak{b}^{\mathfrak{m}}_v$. Then, similarly, the uniform measure γ on C defines a coupling between the measures $\nu_{\mathfrak{m}}$ and $\frac{\mathbf{f}^{\mathfrak{m}} + \delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{m}}| + 1}$, where $\delta_{\mathrm{root}}(x)$ is the indicator that x is the root vertex of $\mathfrak{m}$. As in the quadrangulation case, we have

$$d_{\mathrm{GHP}}\left(\varepsilon \cdot \underline{\mathfrak{m}}\,,\, \varepsilon\kappa_1 \cdot \left(\mathfrak{t}, d_{\mathfrak{t}}, \tfrac{\mathbf{f}^{\mathfrak{m}}}{|\mathbf{f}^{\mathfrak{m}}|}\right)\right) \leqslant \frac{\varepsilon}{2}\,\mathrm{dis}(C; d_{\mathfrak{m}}, \kappa_1 d_{\mathfrak{t}}) + d_{\mathrm{P}}^{(V(\mathfrak{t}), \varepsilon\kappa_1 d_{\mathfrak{t}})}\left(\tfrac{\mathbf{f}^{\mathfrak{m}} + \delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{m}}| + 1}, \tfrac{\mathbf{f}^{\mathfrak{m}}}{|\mathbf{f}^{\mathfrak{m}}|}\right),$$

with the similar bound

$$d_{\mathrm{P}}^{(V(\mathfrak{t}),\varepsilon\kappa_1 d_{\mathfrak{t}})}\left(\frac{\mathbf{f}^{\mathfrak{m}}+\delta_{\mathrm{root}}}{|\mathbf{f}^{\mathfrak{m}}|+1},\frac{\mathbf{f}^{\mathfrak{m}}}{|\mathbf{f}^{\mathfrak{m}}|}\right)\leqslant\frac{2}{|\mathbf{f}^{\mathfrak{m}}|}.$$

The distortion of C is bounded with a very similar argument as above involving Lemma 12 instead of Lemma 13, except that we do not need to introduce $\delta_0,\delta,\widetilde{\delta}$. We leave the details to the reader. This gives for all $(x,v)\in C$ and $(\widetilde{x},\widetilde{v})\in C$, the bound

$$|d_{\mathfrak{m}}(x,\widetilde{x})-\kappa_1 d_{\mathfrak{t}}(v,\widetilde{v})|\leqslant 2\kappa_1+3\Delta(\mathfrak{m})+R(\mathfrak{m},x,\kappa_1)+R(\mathfrak{m},\widetilde{x},\kappa_1),$$

which proves the inequality involving $\mathfrak{m}$ in the statement. □

5.2.9 Proof of Theorem 5

Let $u\geqslant u_C$. Let us first prove the claimed scaling limit for the block-tree $\mathbf{T}_{n,u}$. By Proposition 7, $\mathbf{T}_{n,u}$ has law $\mathrm{GW}(\mu^u,2n)$, where the distribution μ^u has span 2.

Scaling limit of $\mathbf{T}_{n,u}$ for $u>u_C$. If $u>u_C$, then by the third statement of Theorem 1, μ^u is critical and admits a variance $\sigma(u)^2<\infty$. Corollary 15 thus gives the announced scaling limit for $\mathbf{T}_{n,u}$,

$$(2n)^{-1/2}\cdot\underline{\mathbf{T}}_{n,u}\xrightarrow[n\to\infty]{GHP,(d)}\frac{\sqrt{2}}{\sigma(u)}\cdot\mathcal{T}^{(2)}.$$

The expression for $\sigma(u)$ in terms of the generating function B which is given in the statement comes from a straightforward computation from the generating function of μ^u, which by (9) is

$$\sum_{k\geqslant 0}x^k\mu^u(k)=\frac{uB(x^2y(u))+1-u}{uB(y(u))+1-u}.$$

This expression admits the explicit form in terms of u which is given in the statement and explained in Remark 5.

Scaling limit of $\mathbf{T}_{n,u}$ for $u=u_C$. If $u=u_C$, then by the second statement of Proposition 7, μ^{u_C} is critical and satisfies $\mu^{u_C}(\{2j\})\sim\frac{1}{4\sqrt{3\pi}}j^{-5/2}$. Therefore we get the equivalent

$$\mu^{u_C}([x,\infty))=\sum_{2j\geqslant x}\mu^{u_C}(\{2j\})\sim\int_{x/2}^{\infty}\frac{1}{4\sqrt{3\pi}}s^{-5/2}\,\mathrm{d}s=\frac{1}{3}\sqrt{\frac{2}{3\pi}}x^{-3/2}.$$

Therefore, using Corollary 15 with $\theta=3/2$, we get

$$(2n)^{1-2/3}\cdot\underline{\mathbf{T}}_{n,u_C}\xrightarrow[n\to\infty]{GHP,(d)}\left[\frac{\frac{3}{2}-1}{\frac{1}{3}\sqrt{\frac{2}{3\pi}}\Gamma(2-\frac{3}{2})}\right]^{2/3}\cdot\mathcal{T}^{(3/2)}.$$

Using that $\Gamma(1/2)=\sqrt{\pi}$, the constant on the right-hand side simplifies and this translates as announced to

$$\frac{2}{3}(2n)^{-1/3}\cdot\underline{\mathbf{T}}_{n,u_C}\xrightarrow[n\to\infty]{GHP,(d)}\mathcal{T}^{(3/2)}.$$

Restatement of the problem. We let $\alpha = 2$ when $u > u_C$, and $\alpha = 3/2$ when $u = u_C$. We have identified the GHP-limit of $n^{-(\alpha-1)/\alpha} \cdot \underline{\mathbf{T}}_{n,u}$. By Proposition 16, the measured metric spaces

$$n^{-(\alpha-1)/\alpha} \cdot \left(V(\mathbf{T}_{n,u}), d_{\mathbf{T}_{n,u}}, \tfrac{\mathbf{f}^{\mathbf{M}_{n,u}}}{|\mathbf{f}^{\mathbf{M}_{n,u}}|}\right) \quad \text{and} \quad n^{-(\alpha-1)/\alpha} \cdot \left(V(\mathbf{T}_{n,u}), d_{\mathbf{T}_{n,u}}, \tfrac{\mathbf{f}^{\mathbf{Q}_{n,u}}}{|\mathbf{f}^{\mathbf{Q}_{n,u}}|}\right),$$

also converge to the same limit. It remains to compare them in the GHP sense to the measured metric spaces $n^{-(\alpha-1)/\alpha} \cdot \underline{\mathbf{M}}_{n,u}$ and $n^{-(\alpha-1)/\alpha} \cdot \underline{\mathbf{Q}}_{n,u}$ respectively. Let $\kappa_1 = \kappa_u^{\mathrm{map}}$ and $\kappa_2 = \kappa_u^{\mathrm{quad}}$. For the ease of reading, we introduce for $\eta > 0$ the following "bad" events,

$$B^{\mathbf{M}}_{n,\eta} = \left\{ d_{\mathrm{GHP}}\left(n^{-\frac{\alpha-1}{\alpha}} \cdot \underline{\mathbf{M}}\,,\, n^{-\frac{\alpha-1}{\alpha}}\kappa_1 \cdot \left(V(\mathbf{T}), d_{\mathbf{T}}, \tfrac{\mathbf{f}^{\mathbf{M}}}{|\mathbf{f}^{\mathbf{M}}|}\right)\right) \geqslant 2\eta \right\},$$
$$B^{\mathbf{Q}}_{n,\eta} = \left\{ d_{\mathrm{GHP}}\left(n^{-\frac{\alpha-1}{\alpha}} \cdot \underline{\mathbf{Q}}\,,\, n^{-\frac{\alpha-1}{\alpha}}\kappa_2 \cdot \left(V(\mathbf{T}), d_{\mathbf{T}}, \tfrac{\mathbf{f}^{\mathbf{Q}}}{|\mathbf{f}^{\mathbf{Q}}|}\right)\right) \geqslant 2\eta \right\},$$

as well as auxiliary events for $\eta, \delta > 0$,

$$A^{\mathbf{M}}_{1;n,\eta} = \left\{ \exists v \in V(\mathbf{T}),\ R(\mathbf{M}, v, \kappa_1) \geqslant \eta n^{\frac{\alpha-1}{\alpha}} \right\},$$
$$A^{\mathbf{Q}}_{1;n,\eta} = \left\{ \exists v \in V(\mathbf{T}),\ R(\mathbf{Q}, v, \kappa_2) \geqslant \eta n^{\frac{\alpha-1}{\alpha}} \right\},$$
$$A^{\mathbf{M}}_{2;n,\delta} = \left\{ \Delta(\mathbf{M}) \leqslant n^{(1+\delta)^2(\alpha-1)/2\alpha} \right\},$$
$$A^{\mathbf{Q}}_{2;n,\delta} = \left\{ \Delta(\mathbf{Q}) \leqslant n^{(1+\delta)^2(\alpha-1)/2\alpha} \right\},$$
$$A^{\mathbf{M}}_{3;n,\eta} = \left\{ \kappa_1 n^{-\frac{\alpha-1}{\alpha}} + \tfrac{3n^{-\frac{\alpha-1}{\alpha}}}{2}\Delta(\mathbf{M}) + \frac{2}{|\mathbf{f}^{\mathbf{M}}|} \geqslant \eta \right\},$$
$$A^{\mathbf{Q}}_{3;n,\eta} = \left\{ (\kappa_2 + 3) n^{-\frac{\alpha-1}{\alpha}} + \frac{3n^{-\frac{\alpha-1}{\alpha}}}{2}\Delta(\mathbf{Q}) + \frac{4}{|\mathbf{f}^{\mathbf{Q}}|} \geqslant \eta \right\}.$$

With this notation, what we have to show is

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \mathbb{P}_{n,u}(B^{\mathbf{M}}_{n,\eta}) = 0 \quad \text{and} \quad \lim_{\eta\to 0} \limsup_{n\to\infty} \mathbb{P}_{n,u}(B^{\mathbf{Q}}_{n,\eta}) = 0.$$

Using Lemma 23. Thanks to the GHP upper bounds in Lemma 23, we have

$$\mathbb{P}_{n,u}(B^{\mathbf{M}}_{n,\eta}) \leqslant \mathbb{P}_{n,u}(A^{\mathbf{M}}_{1;n,\eta}) + \mathbb{P}_{n,u}(A^{\mathbf{M}}_{3;n,\eta}) \quad \text{and} \quad \mathbb{P}_{n,u}(B^{\mathbf{Q}}_{n,\eta}) \leqslant \mathbb{P}_{n,u}(A^{\mathbf{Q}}_{1;n,\eta}) + \mathbb{P}_{n,u}(A^{\mathbf{Q}}_{3;n,\eta}). \quad (43)$$

Bounding the diameters of the blocks. By Corollary 11, for $\delta > 0$, the maximum diameter of blocks of either $\mathbf{M}_{n,u}$ or $\mathbf{Q}_{n,u}$ is bounded with probability $1 - o(1)$ by $\max(n^{1/6}, W(\mathbf{T}_{n,u})^{(1+\delta)/4})$, where $W(\mathfrak{t})$ denotes the largest degree of $\mathfrak{t}$. By Corollary 15, $W(\mathbf{T}_{n,u})$ is $o\left(n^{(1+\delta)/\alpha}\right)$ in probability. Since $(1+\delta)^2/4\alpha \geqslant 1/6$, what precedes gives that for all $\delta > 0$,

$$\max\big(\Delta(\mathbf{M}_{n,u}), \Delta(\mathbf{Q}_{n,u})\big) = o\left(n^{(1+\delta)^2/4\alpha}\right) \qquad \text{in probability.}$$

Notice that for δ small enough, $(1+\delta)^2/4\alpha < (1+\delta)^2(\alpha-1)/2\alpha$ since $\alpha \geqslant 3/2$. This implies that for all $\delta > 0$ sufficiently small, we have

$$\limsup_{n\to\infty} \mathbb{P}_{n,u}\Big((A^{\mathbf{M}}_{2;n,\delta})^c\Big) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \mathbb{P}_{n,u}\Big((A^{\mathbf{Q}}_{2;n,\delta})^c\Big) = 0.$$

By Lemma 17, the quantities $|\mathbf{f}^{\mathbf{M}}|$ and $|\mathbf{f}^{\mathbf{Q}}|$ are $\Theta(n)$ in probability under $\mathbb{P}_{n,u}$. Therefore, the preceding bound on diameters also implies the following

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(A^{\mathbf{M}}_{3;n,\eta})=0 \quad \text{and} \quad \lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(A^{\mathbf{Q}}_{3;n,\eta})=0.$$

Thanks to (43), it suffices to show that for sufficiently small $\delta>0$,

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(A^{\mathbf{M}}_{1;n,\eta}\cap A^{\mathbf{M}}_{2;n,\delta})=0 \quad \text{and} \quad \lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}(A^{\mathbf{Q}}_{1;n,\eta}\cap A^{\mathbf{Q}}_{2;n,\delta})=0.$$

Bounding the height of $\mathbf{T}_{n,u}$. We have identified above the scaling limit of $\mathbf{T}_{n,u}$ and the appropriate normalization of distances. In particular, $n^{(\alpha-1)/\alpha}\cdot\mathbf{T}_{n,u}$ is tight in the GHP-topology. An immediate consequence is that $n^{-(\alpha-1)/\alpha}H(\mathbf{T}_{n,u})$ is tight, where $H(\mathbf{T}_{n,u})$ is the height of $\mathbf{T}_{n,u}$. In particular, our problem reduces once more to showing that for sufficiently small $\delta>0$,

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}\Big(A^{\mathbf{M}}_{1;n,\eta}\cap A^{\mathbf{M}}_{2;n,\delta}\cap\Big\{H(\mathbf{T})\leqslant\eta^{-1}n^{\frac{(\alpha-1)}{\alpha}}\Big\}\Big)=0$$
$$\text{and}\quad \lim_{\eta\to 0}\limsup_{n\to\infty}\mathbb{P}_{n,u}\Big(A^{\mathbf{Q}}_{1;n,\eta}\cap A^{\mathbf{Q}}_{2;n,\delta}\cap\Big\{H(\mathbf{T})\leqslant\eta^{-1}n^{\frac{(\alpha-1)}{\alpha}}\Big\}\Big)=0. \quad (44)$$

Using the spine decomposition. Fix $\delta>0$, as small as necessary. Let us only treat the term involving $\mathbf{Q}$ in (44), as the expression for $R(\mathfrak{q},x,\kappa_2)$ we used to define the event $A^{\mathbf{Q}}_{1;n,\eta}$ carries more dependence than that of $R(\mathfrak{m},x,\kappa_1)$. Indeed the summands $D_{\mathfrak{q}}(b^{\mathfrak{q}}_{v_i},l_i)$ involve in their definition a global metric property of $\mathfrak{q}$. The case of the term involving $\mathbf{M}$ is similar and easier to deal with.

Recall that by definition, the law $\mathbb{P}_{n,u}$ is the law $\mathbb{P}_u$, conditioned on the event $\{|\mathbf{T}|=n\}$. Since $\mathbb{P}_u(|\mathbf{T}|=n)$ decays polynomially by Corollary 15, we may get rid of the conditioning if the unconditional versions of the probabilities we wish to bound decay sufficiently fast. Namely, it suffices to prove that for all $\eta>0$, the following (unconditional) probability is stretched-exponential in n

$$\mathbb{P}_u\Big(A^{\mathbf{Q}}_{1;n,\eta}\cap A^{\mathbf{Q}}_{2;n,\delta}\cap\{H(\mathbf{T})\leqslant\eta^{-1}n^{\frac{(\alpha-1)}{\alpha}}\}\Big). \quad (45)$$

By a union bound and then by Proposition 18, one can bound this by

$$\begin{aligned}
&\mathbb{E}_u\left[\sum_{v\in V(\mathbf{T})}\mathbb{1}_{\left\{R(\mathbf{Q},v,\kappa_2)\geqslant\eta n^{\frac{\alpha-1}{\alpha}}\right\}}\mathbb{1}_{\left\{H(\mathbf{T})\leqslant\eta^{-1}n^{\frac{\alpha-1}{\alpha}}\right\}}\mathbb{1}_{\{A^{\mathbf{Q}}_{2;n,\delta}\}}\right]\\
&=\sum_{h\geqslant 1}\widehat{\mathbb{P}}_{u,h}\left(\{R(\mathbf{Q},V_\star,\kappa_2)\geqslant\eta n^{\frac{\alpha-1}{\alpha}}\}\cap\{H(\mathbf{T})\leqslant\eta^{-1}n^{\frac{\alpha-1}{\alpha}}\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&=\sum_{h=1}^{\lfloor\eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\widehat{\mathbb{P}}_{u,h}\left(\{R(\mathbf{Q},V_\star,\kappa_2)\geqslant\eta n^{\frac{\alpha-1}{\alpha}}\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&=\sum_{h=1}^{\lfloor\eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\widehat{\mathbb{P}}_{u,h}\left(\left\{\max_{0\leqslant i<h}\left|\sum_{j=i}^{h-1}\left(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{v_j},L_j)-\kappa_2\right)\right|\geqslant\eta n^{\frac{\alpha-1}{\alpha}}\right\}\cap A^{\mathbf{Q}}_{2;n,\delta}\right)\\
&\leqslant\sum_{h=1}^{\lfloor\eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\left[\widehat{\mathbb{P}}_{u,h}\left(\max_{0\leqslant i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_j},L_j)-\kappa_2\right)\geqslant\eta n^{\frac{\alpha-1}{\alpha}}\right)\right.
\end{aligned}$$

$$+\widehat{\mathbb{P}}_{u,h}\left(\max_{0\leqslant i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(\kappa_2-D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{V_j},L_j)\right)\geqslant \eta n^{\frac{\alpha-1}{\alpha}}\right)\Bigg],$$

where

$$\psi_{n,\delta}(x)=x\mathbb{1}_{\left\{x\leqslant \max(\kappa_2,n^{(1+\delta)^2(\alpha-1)/2\alpha})\right\}}.$$

The last inequality may require some explanations. First we apply a union bound with respect to the sign of the expression under the absolute value. Then we use the control that $A^{\mathbf{Q}}_{2;n,\delta}$ offers on $\Delta(\mathbf{Q})$ the maximum diameter of blocks of $\mathbf{Q}$, and the positivity of the distances $D_{\mathbf{Q}}(\mathfrak{b}^{\mathbf{Q}}_{v_j})$, to insert an indicator function. Hence the appearance of $\psi_{n,\delta}$.

Reducing to a large deviations event with truncated variables. We let $\left(\widehat{\xi}_{u,j}\,,\,\widehat{\mathbf{B}}^{\mathrm{quad}}_{u,j}\,,\,U_j\right)_{j\geqslant 0}$ be an i.i.d. sequence of copies of the triple $\left(\widehat{\xi}_u\,,\,\widehat{\mathbf{B}}^{\mathrm{quad}}_{u}\,,\,U\right)$. We also let $X_j=D\big(\widehat{\mathbf{B}}^{\mathrm{quad}}_{u,j},U_j\big)-\kappa_2$. Then by Lemma 20, the arguments of the function $\psi_{n,\delta}$ that appear in the last upper bound we obtained, are actually i.i.d. and have joint law under $\widehat{\mathbb{P}}_{u,h}$ the law of $(X_j)_{0\leqslant j<h}$. Therefore this last upper bound is equal to

$$\sum_{h=1}^{\lfloor \eta^{-1}n^{\frac{\alpha-1}{\alpha}}\rfloor}\left[P\left(\max_{0\leqslant i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(X_j\right)\geqslant \eta n^{\frac{\alpha-1}{\alpha}}\right)+P\left(\max_{0\leqslant i<h}\sum_{j=i}^{h-1}\psi_{n,\delta}\left(-X_j\right)\geqslant \eta n^{\frac{\alpha-1}{\alpha}}\right)\right].$$

Since the sequence $(X_j)_{0\leqslant j<h}$ is i.i.d., we re-order the terms of the two sums which appear inside the probabilities in the last display, so that they run on indices $j=1,\dots,i$. Hence, if we set $h_n=n^{(\alpha-1)/\alpha}$, then we can bound the last display by

$$\eta^{-1}h_n\left[P\left(\max_{0\leqslant i<\eta^{-1}h_n}\sum_{j=1}^{i}\psi_{n,\delta}\left(X_j\right)\geqslant \eta h_n\right)+P\left(\max_{0\leqslant i<\eta^{-1}h_n}\sum_{j=1}^{i}\psi_{n,\delta}\left(-X_j\right)\geqslant \eta h_n\right)\right].$$

Using the moderate deviations estimate. Let $\gamma=\gamma(\delta)=(1+\delta)^2/2$. Then $(h_n)^\gamma\geqslant\kappa_2$ for n large, so

$$\psi_{n,\delta}(x)=x\mathbb{1}_{\{x\leqslant\max(\kappa_2,(h_n)^\gamma)\}}=x\mathbb{1}_{\{x\leqslant(h_n)^\gamma\}}.$$

Let us check that the choice of $\kappa_2=\kappa_u^{\mathrm{quad}}$, the latter quantity being defined in (30), makes the variables (X_j) centered. Conditionally on the event $\{|\widehat{\mathbf{B}}^{\mathrm{quad}}_u|=k\}$, the variable $\widehat{\mathbf{B}}^{\mathrm{quad}}_u$ is a uniform simple quadrangulation with k edges, and U is uniform in $\{1,\dots,2k\}$. Therefore it holds that

$$\begin{aligned}E\left[D\big(\widehat{\mathbf{B}}^{\mathrm{quad}}_{u,j},U_j\big)\right]&=\sum_{k\geqslant 1}P\left(|\widehat{\mathbf{B}}^{\mathrm{quad}}_u|=k\right)E\left[D\big(\widehat{\mathbf{B}}^{\mathrm{quad}}_u,U\big)\mid|\widehat{\mathbf{B}}^{\mathrm{quad}}_u|=k\right]\\&=\sum_{k\geqslant 1}\hat{\mu}^u(2k)\mathcal{D}^{\mathrm{quad}}_k=\kappa_2,\end{aligned}$$

where we used successively the definition of $(\mathfrak{b},l)\mapsto D(\mathfrak{b},l)$ in Lemma 20, the definition of $\mathcal{D}^{\mathrm{quad}}_k$ after (30), and the definition of $\kappa_2=\kappa_u^{\mathrm{quad}}$ in (30). Therefore, the i.i.d. variables (X_j) are indeed

centered. Now by Proposition 21, they possess moments of order β for all $1 \leqslant \beta < 2$. Since for δ sufficiently small we have $\gamma < 1$, Proposition 22 yields that

$$P\left(\max_{0\leqslant i<\eta^{-1}h_n} \sum_{j=1}^{i} X_j \mathbb{1}_{\{X_j\leqslant (h_n)^\gamma\}} \geqslant \eta h_n\right)$$

is stretched-exponential as $n \to \infty$, and the same holds when replacing (X_j) by $(-X_j)$.

This proves that for each $\eta > 0$, the probability (45) is indeed stretched-exponential in n, and concludes the proof. □

5.3 Scaling limit of the quadrangulations in the subcritical case

Let us finally identify the scaling limit of the quadrangulation $\underline{\mathbf{Q}}_{n,u}$ when $u < u_C$.

5.3.1 Statement of the result

Denote by $\mathcal{S} = (\mathcal{S}, D^*, \lambda)$ the *Brownian sphere*, also known as the *Brownian Map*. One may take Proposition 24 below as a definition.

Theorem 6. *Assume $u < u_C = 9/5$. We have the following convergence in distribution for the* Gromov-Hausdorff-Prokhorov *topology*

$$\left(\frac{9(3+u)}{8(9-5u)}\right)^{1/4} n^{-1/4} \cdot \underline{\mathbf{Q}}_{n,u} \xrightarrow[n\to\infty]{(d),\mathrm{GHP}} \mathcal{S}.$$

In the case $u = 1$, one recovers the Brownian sphere as the scaling limit of uniform quadrangulations with n faces, which has been proven in [LG13] and [Mie13]. It is also the scaling limit of uniform *simple* quadrangulations with n faces, which was proven in [ABA17]. The latter corresponds informally to the case $u \to 0$.

We emphasize that those results, and especially the one of [ABA17], serve as an input in our proof and we do not provide a new proof of them. Accordingly, let us precisely state the latter result, so that we can use it in the subsequent proof.

Proposition 24 ([ABA17])**.** *Uniform simple quadrangulations with k faces admit the Brownian sphere as scaling limit, with the following normalization*

$$\left(\frac{3}{8k}\right)^{1/4} \cdot B_k^{\mathrm{quad}} \xrightarrow[k\to\infty]{(d),\mathrm{GHP}} \mathcal{S}.$$

This is precisely the result [ABA17, Theorem 1.1], restricted to the case of simple quadrangulations. Notice that in their result, the scaling limit is stated in terms of M_n, a uniform simple quadrangulation with n vertices, not faces. This is not a problem since by Euler's formula, a quadrangulation has n vertices if and only if it has $n-2$ faces. Therefore B_k^{quad} has the same law as M_{k+2}.

Note that Theorem 6 only deals with the quadrangulation $\underline{\mathbf{Q}}_{n,u}$, but not the map $\underline{\mathbf{M}}_{n,u}$. Let us detail what would be needed to obtain a similar statement for $\underline{\mathbf{M}}_{n,u}$.

- To obtain a Gromov-Hausdorff scaling limit, the missing ingredient is the equivalent for 2-connected maps of the result of [ABA17], that is to say GH(P) convergence of uniform 2-connected maps with n edges, rescaled by a constant times $n^{-1/4}$, to the Brownian sphere.

- In order to strengthen this to GHP convergence when the map is equipped with the uniform measure on vertices, one would need the above mentioned convergence of 2-connected maps, but in the GHP sense. It would also require a way to compare, in the Prokhorov sense, the degree-biased measure on vertices of $\underline{\mathbf{M}}_{n,u}$, and the uniform measure. For quadrangulations on the other hand, this comparison can be done using [ABW17, Lemma 5.1].

The paper [ABW17] makes precise the relationship between the convergence of **uniform** quadrangulations with n faces [LG13, Mie13], and the convergence of **simple uniform** quadrangulations with n faces [ABA17]. It is shown that a quadrangulation sampled uniformly among those which have size n and whose biggest block has size $k(n) \sim cn$ with an adequate $c > 0$, converges jointly with said biggest block to the Brownian sphere, in the GHP sense.

The proof of Gromov-Hausdorff convergence for these quadrangulations amounts to showing that pendant submaps that are grafted on the macroscopic block have negligible diameter, that is $o\left(n^{1/4}\right)$, which is done by [ABW17, Proposition 1.12]. The strategy of proof is not directly applicable here, since it uses an *a priori* diameter bound on the pendant submaps, which we do not have for general u. As explained in what follows, it is sufficient to have an *a priori* diameter bound on single blocks themselves, which is why we need Proposition 10. To strengthen GH convergence to GHP convergence however, we use the same arguments as those exposed in [ABW17] modulo some technical details.

5.3.2 Sketch of the proof

On the combinatorics side, Theorem 2 characterizes the phase $u < u_C$ by a condensation phenomenon: when n is large, there is precisely one block of linear size, while others have size $O(n^{2/3})$. This theorem is stated for a map with law $\mathbb{P}_{n,u}$, that is the law of $\mathbf{M}_{n,u}$, but by Section 2.5, Tutte's bijection commutes with the block decomposition, so that the same happens for $\mathbf{Q}_{n,u}$.

On the metric side, there is not much more going on. The block-tree is subcritical in this phase by Theorem 1 and therefore has small height. Combining this with the $O(n^{2/3})$ bound on the size of non-macroscopic blocks, and the deviation estimate of Proposition 10 on diameters of blocks, we get that $\mathbf{Q}_{n,u}$ is approximately equal to its largest block, in the Gromov-Hausdorff sense in the scale $n^{1/4}$. This argument is rather general and should be easy to adapt to other models of graphs or maps with a block-tree decomposition under a condensation regime.

In order to strengthen this convergence to one in the Gromov-Hausdorff-Prokhorov sense, we use the rather general result [ABW17, Corollary 7.2], by comparing the mass measure on vertices with a projection on the macroscopic block, which is modulo some technical details an exchangeable vector on the edges where the pendant submaps are attached. This corollary tells that this random measure is well-approximated by its expectation, which is uniform on the edges of the macroscopic block, or equivalently that it is degree-biased on its vertices. The last part of the argument is specific to quadrangulations, for which we can compare the degree-biased and the uniform measure on vertices by [ABW17, Lemma 6.1].

5.3.3 Comparison of a quadrangulation and its biggest block

Let us introduce some notation. Let v° be the vertex of $\mathfrak{t}$ with largest outdegree, choosing one arbitrarily if there are several, and let $\mathfrak{q}^\circ = \mathfrak{b}^{\mathfrak{q}}_{v^\circ}$. Denote by $\mathfrak{t}[v]$ the subtree of descendants of a node v in $\mathfrak{t}$, rooted at v. For an edge e of $\mathfrak{q}^\circ = \mathfrak{b}^{\mathfrak{q}}_{v^\circ}$, the block-tree decomposition associates to it a vertex v, so that we can denote by $\mathfrak{q}[e]$ the quadrangulation whose block-tree decomposition is $(\mathfrak{b}^{\mathfrak{q}}_w)_{w\in\mathfrak{t}[v]}$. Recall that by convention, if v is a leaf then $\mathfrak{q}[e]$ is the edge map, with 2 vertices and 1 edge, the edge e. Write also $\mathfrak{q}^+$ for the quadrangulation whose block decomposition is $(\mathfrak{b}^{\mathfrak{q}}_v)_{v\in\mathfrak{t}[v^\circ]}$. In particular, $\mathfrak{q}^\circ$ is the simple core of $\mathfrak{q}^+$, and its other blocks are the blocks of the pendant subquadrangulations $(\mathfrak{q}[e])_{e\in E(\mathfrak{q}^\circ)}$. Finally, let $\pi^{\mathfrak{q}^+}_{\mathfrak{q}^\circ}$ be the probability measure on vertices of $\mathfrak{q}^\circ$ obtained by projection of the contribution to $\nu_{\mathfrak{q}}$ of each pendant map $(\mathfrak{q}[e])_{e\in E(\mathfrak{q}^\circ)}$ to the biggest block $\mathfrak{q}^\circ$. More formally, for each edge e of $\mathfrak{q}^\circ$, let $\{e^+, e^-\}$ be its extremities. Then,

$$\pi^{\mathfrak{q}^+}_{\mathfrak{q}^\circ} = \frac{1}{|V(\mathfrak{q}^+)| - |V(\mathfrak{q}^\circ)|} \sum_{e\in E(\mathfrak{q}^\circ)} \big(|V(\mathfrak{q}[e])| - 2\big)\,\big(\tfrac{1}{2}\delta_{e^-} + \tfrac{1}{2}\delta_{e^+}\big).$$

Observe that $V(\mathfrak{q}^\circ)$ shares exactly two elements with each $(V(\mathfrak{q}[e]))_{e\in E(\mathfrak{q}^\circ)}$, when those vertex-sets are naturally embedded in $V(\mathfrak{q})$. Hence the last display indeed defines a probability measure.

Lemma 25. *For any $\varepsilon > 0$, it holds that*

$$d_{\mathrm{GHP}}(\varepsilon\cdot\underline{\mathfrak{q}}, \varepsilon\cdot\underline{\mathfrak{q}}^\circ) \leqslant 2r_{\mathrm{GH}} + r_{\mathrm{P}} + \big(1 - \tfrac{|V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|}\big) d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}\Big(\pi^{\mathfrak{q}^+}_{\mathfrak{q}^\circ}, \nu_{\mathfrak{q}^\circ}\Big),$$

where

$$r_{\mathrm{GH}} = 2\varepsilon H(\mathfrak{t}) \max_{v\neq v^\circ} \mathrm{diam}(\mathfrak{b}^{\mathfrak{q}}_v) \quad \text{and} \quad r_{\mathrm{P}} = \frac{2\big|V(\mathfrak{q})\setminus V(\mathfrak{q}^+)\big|}{|V(\mathfrak{q})|}.$$

Proof. There are successive comparisons to be made for the GHP distance.

Metric comparison. The term r_{GH} bounds how distant the spaces $\varepsilon\cdot\mathfrak{q}$, $\varepsilon\cdot\mathfrak{q}^+$ and $\varepsilon\cdot\mathfrak{q}^\circ$ are, from a metric point of view, *i.e.* in the GH sense. Recall that we can see $\mathfrak{q}$ and $\mathfrak{q}^+$ as their biggest block $\mathfrak{q}^\circ$, together with some maps attached to it. Therefore one needs to bound the maximal diameter of the attached maps. We use a brutal bound on the diameter of the non-macroscopic blocks by their maximal diameter, together with a bound on the number of consecutive blocks in the attached maps. This number is bounded by $\mathrm{diam}(\mathfrak{t}) \leqslant 2H(\mathfrak{t})$. Therefore the maximal diameter of attached maps in $\varepsilon\cdot\mathfrak{q}$ or $\varepsilon\cdot\mathfrak{q}^+$ is bounded by

$$r_{\mathrm{GH}} := 2\varepsilon H(\mathfrak{t}) \max_{v\neq v^\circ} \mathrm{diam}(\mathfrak{b}^{\mathfrak{q}}_v).$$

In particular, take the correspondence B_1 on $V(\mathfrak{q}) \times V(\mathfrak{q}^+)$ such that $x \in V(\mathfrak{q})$ is in correspondence with only itself if it belongs to $V(\mathfrak{q}^+)$, or otherwise with both endpoints of the root-edge of $\mathfrak{q}^+$ if it belongs to $V(\mathfrak{q})\setminus V(\mathfrak{q}^+)$. The uniform measure on B_1 is a coupling between $\nu_{\mathfrak{q}}$ and some measure μ^+ on $V(\mathfrak{q}^+)$. One therefore gets, using the triangle inequality and (25),

$$d_{\mathrm{GHP}}(\varepsilon\cdot\underline{\mathfrak{q}}, \varepsilon\cdot\underline{\mathfrak{q}}^+) \leqslant r_{\mathrm{GH}} + d_{\mathrm{P}}^{(V(\mathfrak{q}^+), \varepsilon d_{\mathfrak{q}})}(\mu^+, \nu_{\mathfrak{q}^+}). \tag{46}$$

Similarly, take the correspondence B_2 on $V(\mathfrak{q}^+) \times V(\mathfrak{q}^\circ)$ such that $x \in V(\mathfrak{q}^+)$ is in correspondence with only itself if it belongs to $V(\mathfrak{q}^\circ)$, or otherwise with both endpoints $\{e^+, e^-\}$ of the root-edge of $\mathfrak{q}[e]$ if x belongs to $V(\mathfrak{q}[e]) \setminus \{e^+, e^-\}$ for some edge $e \in E(\mathfrak{q}^\circ)$. Then the uniform measure on B_2 is a coupling between $\nu_{\mathfrak{q}^+}$ and some measure μ° on $V(\mathfrak{q}^+)$. We get as above

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}^+, \varepsilon \cdot \underline{\mathfrak{q}}^\circ) \leqslant r_{\mathrm{GH}} + d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}(\mu^\circ, \nu_{\mathfrak{q}^\circ}). \tag{47}$$

Comparing the uniform measures on vertices of $\mathfrak{q}$ and $\mathfrak{q}^+$. Observe that $\nu_{\mathfrak{q}^+}$ is the counting measure on $V(\mathfrak{q}^+)$ renormalized to a probability distribution, while μ^+ is the renormalized version of the same counting measure but with additional mass

$$m := |V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)| - 2,$$

the latter being split equally on the endpoints of the root-edge of $\mathfrak{q}^+$. Elementarily, this yields a total variation bound, as follows

$$d_{\mathrm{TV}}(\mu^+, \nu_{\mathfrak{q}^+}) \leqslant \frac{2m}{V(\mathfrak{q})} \leqslant \frac{2|V(\mathfrak{q}) \setminus V(\mathfrak{q}^+)|}{V(\mathfrak{q})} =: r_{\mathrm{P}}.$$

Since the Prokhorov distance is bounded by the total variation distance, we have

$$d_{\mathrm{P}}^{(V(\mathfrak{q}^+), \varepsilon d_{\mathfrak{q}})}(\mu^+, \nu_{\mathfrak{q}^+}) \leqslant r_{\mathrm{P}}. \tag{48}$$

Comparing the uniform measures on vertices of $\mathfrak{q}^+$ and $\mathfrak{q}^\circ$. From the definition of $\pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}$ and from the following partitioning, under the natural embedding of the vertex-sets in $V(\mathfrak{q})$,

$$V(\mathfrak{q}^+) = V(\mathfrak{q}^\circ) \bigsqcup_{e \in E(\mathfrak{q}^\circ)} V(\mathfrak{q}[e]) \setminus \{e^+, e^-\},$$

observe that the measure μ° obtained from the correspondence B_2 above decomposes as follows

$$\mu^\circ = \tfrac{|V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|} \nu_{\mathfrak{q}^\circ} + \tfrac{|V(\mathfrak{q}^+)| - |V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|} \pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}.$$

In particular, we obtain from (26) that

$$d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}(\mu^\circ, \nu_{\mathfrak{q}^\circ}) = \left(1 - \tfrac{|V(\mathfrak{q}^\circ)|}{|V(\mathfrak{q}^+)|}\right) d_{\mathrm{P}}^{(V(\mathfrak{q}^\circ), \varepsilon d_{\mathfrak{q}})}\left(\pi_{\mathfrak{q}^\circ}^{\mathfrak{q}^+}, \nu_{\mathfrak{q}^\circ}\right). \tag{49}$$

Concluding the proof. By the triangle inequality, we have

$$d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^\circ) \leqslant d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}, \varepsilon \cdot \underline{\mathfrak{q}}^+) + d_{\mathrm{GHP}}(\varepsilon \cdot \underline{\mathfrak{q}}^+, \varepsilon \cdot \underline{\mathfrak{q}}^\circ).$$

Using (46) and (48) to bound the first term, and (47) and (49) to bound the second one, we get the claimed inequality.

□

5.3.4 Exchangeable decorations

We aim to use Addario-Berry & Wen's argument for [ABW17, Lemma 6.2] which tells that for exchangeable attachments of mass on edges of Q_n, a quadrangulation with n faces sampled uniformly, the resulting measure on Q_n is asymptotically close to the uniform measure on vertices, in the sense of the Prokhorov distance on $n^{-1/4} \cdot Q_n$. They use the following ingredients:

1. A concentration inequality [ABW17, Lemma 5.2] which compares the measure with exchangeables attachments of mass on edges, to the degree-biased measure on vertices.
2. A Prokhorov comparison [ABW17, Lemma 5.1] between the degree-biased and uniform measure on vertices of a quadrangulation.
3. GHP convergence of $n^{-1/4} \cdot \underline{Q}_n$ to the Brownian sphere.
4. Properties of the Brownian sphere such as compacity and re-rooting invariance.

The first ingredient is rather general and actually stated for any graph in [ABW17, Lemma 5.3]. We will ever-so-slightly adapt its proof since there is a double edge in their setting which we do not have, and the mass is not projected on vertices in the exact same way. The second ingredient is specific to quadrangulations and one may need different arguments to compare the degree-biased and uniform measures for other classes of maps.

Let us state which result we extract for our purpose from Addario-Berry & Wen's paper. For $\mathbf{n} = (\mathbf{n}(e))_{e\in E(G)}$ a family of nonnegative numbers indexed by edges of a graph G, we denote its p-norm for $p \geqslant 1$ by

$$|\mathbf{n}|_p := \left(\sum_{e\in E(G)} (\mathbf{n}(e))^p \right)^{1/p}.$$

Then define the following measure on $V(G)$:

$$\mu_G^{\mathbf{n}} := \frac{1}{|\mathbf{n}|_1} \sum_{e\in E(G)} \mathbf{n}(e)\big(\tfrac{1}{2}\delta_{e^+} + \tfrac{1}{2}\delta_{e^-}\big),$$

with $\{e^+, e^-\}$ the set of endpoints of the edge e. Notice that this definition is slightly different from that of $\nu_G^{\mathbf{n}}$ in [ABW17, Section 5], because the mass of an edge is projected uniformly and independently on either of its enpoints in their case, while we deterministically split this mass on both endpoints. This does not change much except that we find it easier to work with. One of their results translates as the following.

Proposition 26 ([ABW17, Corollary 6.2]). *Let $Q_k = B_k^{\mathrm{quad}}$, which is a simple quadrangulation with k faces, sampled uniformly. Consider for each $k \geqslant 1$, a random family $\mathbf{n}_k = (\mathbf{n}_k(e))_{e\in E(Q_k)}$ of nonnegative numbers, such that conditionally on Q_k it is an exchangeable family. Assume that $|\mathbf{n}_k|_2/|\mathbf{n}_k|_1 \to 0$ in probability as $k \to \infty$. Then there holds the convergence in probability*

$$d_{\mathrm{P}}^{(V(Q_k),\varepsilon_k d_{Q_k})}\Big(\mu_{B_k}^{\mathbf{n}_k}, \nu_{Q_k}\Big) \xrightarrow[k\to\infty]{\mathbb{P}} 0,$$

where ν_{Q_k} is the uniform measure on vertices of Q_k and $\varepsilon_k = k^{-1/4}$.

This is the statement of [ABW17, Corollary 6.2], adapted to our setting. The proof goes *mutatis mutandi*, except for an adjustement in the concentration inequality [ABW17, Lemma 5.3], which we adapt below in Lemma 27.

Lemma 27 ([ABW17, Lemma 5.3])**.** *Let G be a graph and $\mathbf{n} = (\mathbf{n}(e))_{e\in G}$ a random and exchangeable family of nonnegative numbers with $|\mathbf{n}|_2 > 0$ almost surely. Then for any $V \subset V(G)$, and any $t > 0$,*

$$\mathbb{P}\left(\left|\mu_G^{\mathbf{n}}(V) - \nu_G(V)\right| > \frac{2t}{|\mathbf{n}|_1} \;\middle|\; |\mathbf{n}|_2\right) \leqslant 2\exp\left(-\frac{2t^2}{|\mathbf{n}|_2^2}\right).$$

The proof goes the same way as that of [ABW17, Lemma 5.3], except that we do not have a double edge here, and the mass on edges is projected deterministically on vertices in our case, instead of randomly. The reader may notice that there is an extra term inside the probability in their lemma. This term accounts for the double edge, which we do not have here. The same line of arguments still works though. Indeed, we have

$$\mu_G^{\mathbf{n}}(V) = \sum_{e\in E(G[V])} \frac{\mathbf{n}(e)}{|\mathbf{n}|_1} + \frac{1}{2}\sum_{e\in\partial_e V} \frac{\mathbf{n}(e)}{|\mathbf{n}|_1},$$

with $G[V]$ the induced-graph on V by G, and $\partial_e V$ the subset of the edges of V who have only one endpoint which belongs to V. By exchangeability, we have the expectation

$$\mathbb{E}\left[\sum_{e\in E(G[V])} \mathbf{n}(e) + \frac{1}{2}\sum_{e\in\partial_e V} \mathbf{n}(e) \;\middle|\; |\mathbf{n}|_1\right] = |\mathbf{n}|_1 \frac{|E(G[V])|}{|E(G)|} + |\mathbf{n}|_1 \frac{\frac{1}{2}|\partial_e V|}{|E(G)|} = |\mathbf{n}|_1 \nu_G(V).$$

The last equality holds because the degree biased-measure counts twice each edge of $G[V]$, since this edge appears in the degree of both its endpoints, while the edges of $\partial_e V$ are only counted once, in the degree on the only one of its endpoints which is in V.

Then one concludes as in the proof of [ABW17, Lemma 5.3], by a Hoeffding-type bound for exchangeable vectors.

5.3.5 Proof of Theorem 6

Scaling limit of the biggest block. By Proposition 3 and Proposition 4, the biggest block of $\mathbf{Q}_{n,u}$, whose size we denote $C(n,u)$, is a simple quadrangulation sampled uniformly with size $C(n,u)$. Also by Theorem 2, this size is asymptotically in probability,

$$C(n,u) = (1 - E(u))n + O_{\mathbb{P}}(n^{2/3}) = \frac{9-5u}{3(3+u)}n + O_{\mathbb{P}}(n^{2/3}).$$

By conditioning on $C(n,u)$ and using Proposition 24, we therefore get the following GHP scaling limit for the biggest block

$$\left(\frac{3}{8C(n,u)}\right)^{1/4} \cdot \underline{\mathbf{Q}}^{\circ}_{n,u} \xrightarrow[n\to\infty]{(d),\mathrm{GHP}} \mathcal{S},$$

which by the preceding equivalent in probability for $C(n,u)$ reduces to

$$\left(\frac{9(3+u)}{8(9-5u)}\right)^{1/4} n^{-1/4} \cdot \underline{\mathbf{Q}}^{\circ}_{n,u} \xrightarrow[n\to\infty]{(d),\mathrm{GHP}} \mathcal{S}.$$

GHP comparison of $\mathbf{Q}_{n,u}$ with its biggest block. By the preceding scaling limit, and the use of Lemma 25 with $\mathfrak{q} = \mathbf{Q}_{n,u}$ and $\varepsilon = n^{-1/4}$, the proof of the theorem reduces to showing the convergence to 0 in probability of the following quantities

$$\begin{aligned} r_{\mathrm{GH}} &:= \frac{2}{n^{1/4}} H(\mathbf{T}_{n,u}) \max_{v \neq v^\circ} \mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) \\ r_{\mathrm{P}} &:= \frac{2\big|V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}_{n,u}^+)\big|}{|V(\mathbf{Q}_{n,u})|} \\ d_P &:= d_{\mathrm{P}}^{(V(\mathbf{Q}_{n,u}^\circ), \varepsilon_n d_{\mathbf{Q}_{n,u}})}\Big(\pi_{\mathbf{Q}_{n,u}^\circ}^{\mathbf{Q}_{n,u}^+}, \nu_{\mathbf{Q}_{n,u}^\circ}\Big), \end{aligned}$$

where $\varepsilon_n = n^{-1/4}$.

Bounding r_{GH}. By Theorem 2, the second-biggest block of $\mathbf{Q}_{n,u}$ has size $O(n^{2/3})$ in probability. Combining this with Corollary 11, one gets for all $\delta > 0$ the bound in probability

$$\max_{v \neq v^\circ} \mathrm{diam}(\mathfrak{b}_v^{\mathbf{Q}_{n,u}}) = o\left(n^{(1+\delta)/6}\right).$$

Also, by Theorem 1, $\mathbf{T}_{n,u}$ is a *non-generic subcritical* Galton-Watson tree conditioned to have $2n+1$ vertices, in the terminology of [Kor15]. We may therefore use [Kor15, Theorem 4] to get for all $\delta > 0$ the bound in probability

$$H(\mathbf{T}_{n,u}) = o\left(n^{\delta}\right).$$

Combining the two preceding estimates, we get in probability

$$r_{\mathrm{GH}} = o\left(n^{-\frac{1}{4}+\delta+\frac{(1+\delta)}{6}}\right) \xrightarrow[n\to\infty]{} 0,$$

provided that we chose $\delta > 0$ small enough so that $\delta + (1+\delta)/6 < 1/4$.

Bounding r_{P}. First, notice that since $\mathbf{Q}_{n,u}$ is a quadrangulation we have

$$|V(\mathbf{Q}_{n,u})| = |E(\mathbf{Q}_{n,u})| = 2n,$$

and by the block-tree decomposition which puts in correspondence edges of $\mathbf{Q}_{n,u}$ and edges of $\mathbf{T}_{n,u}$, we also have

$$\big|V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}_{n,u}^+)\big| = \big|E(\mathbf{Q}_{n,u})\big| - \big|E(\mathbf{Q}_{n,u}^+)\big| = \big|E(\mathbf{T}_{n,u})\big| - \big|E(\mathbf{T}_{n,u}^+)\big| = \big|E(\mathbf{T}_{n,u} \setminus \mathbf{T}_{n,u}^+)\big|.$$

Therefore we have to bound the size of the subtree $\mathbf{T}_{n,u} \setminus \mathbf{T}_{n,u}^+$. A moment of thought shows that it is bounded by

$$U_{\rightarrow}(\mathbf{T}_{n,u}) + U_{\leftarrow}(\mathbf{T}_{n,u}),$$

where $U_{\rightarrow}(\mathfrak{t})$ is the index in lexicographical order of the vertex with largest degree of the tree $\mathfrak{t}$, and $U_{\leftarrow}(\mathfrak{t})$ is the index in reverse lexicographical order of that same vertex. Now, [Kor15, Theorem 2] shows that $(U_{\rightarrow}(\mathbf{T}_{n,u}))_{n\geqslant 1}$ is a tight sequence. Since $U_{\leftarrow}(\mathbf{T}_{n,u})$ has the same law as $U_{\rightarrow}(\mathbf{T}_{n,u})$, the respective sequence is also tight. All in all, we get in probability

$$r_{\mathbf{P}} = O(1/n) \xrightarrow[n\to\infty]{} 0.$$

Bounding d_{P}. Notice that

$$d_P = d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\pi^{\mathbf{Q}^+_{n,u}}_{\mathbf{Q}^\circ_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right) = d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\mu^{\mathbf{n}}_{\mathbf{Q}_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right),$$

where $\mathbf{n} = \mathbf{n}_{n,u}$ is the family of nonnegative numbers defined by

$$\forall e \in E(\mathbf{Q}^\circ_{n,u}), \quad \mathbf{n}(e) = |V(\mathbf{Q}_{n,u}[e])| - 2.$$

Let us argue that conditionally on $\mathbf{Q}^\circ_{n,u}$, this family $\mathbf{n}$ is exchangeable. Recall that $\mathbf{Q}_{n,u}$ has the law of $\mathbf{Q}$ under $\mathbb{P}_u$, conditioned to the event $\{|\mathbf{Q}| = n\}$. By the symmetries of the Galton-Watson law and Proposition 7, the family

$$(|V(\mathbf{Q}[e])| - 2)_{e\in E(\mathbf{Q}^\circ)} = (|E(\mathbf{T}[v_e])| - 2)_{e\in E(\mathbf{Q}^\circ)}$$

is i.i.d. conditionally on $\mathbf{Q}^\circ$, where v_e is the child of v° that the block-tree decomposition associates to e. In particular, this family is exchangeable. Since the event $\{|\mathbf{Q}| = n\}$ is invariant by each permutation of the subtrees attached to the node v° with their respective blocks, the above family stays exchangeable when conditioning by this event. Therefore $\mathbf{n}$ is indeed exchangeable.

Now, [Kor15, Corollary 1] tells that the subtrees $(\mathbf{T}[v_e])_{e\in\mathbf{Q}^\circ_{n,u}}$ have size $O(n^{2/3})$ in probability, uniformly in the edge e. We thus get that

$$|\mathbf{n}|_2 = O\left(\sqrt{n^{5/3}}\right).$$

On the other hand, we have in probability

$$|\mathbf{n}|_1 = |V(\mathbf{Q}_{n,u}) \setminus V(\mathbf{Q}^+_{n,u})| \sim cn,$$

for some constant $c > 0$. Hence, in probability

$$\frac{|\mathbf{n}|_2}{|\mathbf{n}|_1} = O\left(n^{-1/6}\right) \xrightarrow[n\to\infty]{} 0.$$

All the hypotheses of Proposition 26 have been checked, so that we may apply it, after conditioning by the size of $\mathbf{Q}^\circ_{n,u}$, since conditionally on its size k it is a uniform simple quadrangulation of size k. We obtain in probability

$$d_{\mathrm{P}}^{(V(\mathbf{Q}^\circ_{n,u}),\varepsilon_n d_{\mathbf{Q}_{n,u}})}\left(\mu^{\mathbf{n}}_{\mathbf{Q}_{n,u}}, \nu_{\mathbf{Q}^\circ_{n,u}}\right) \xrightarrow[n\to\infty]{} 0.$$

Hence, d_{P} also tends to 0 in probability and this concludes the proof. □

6 Concluding remarks and perspectives

We have exhibited a phase transition phenomenon for two closely related models of random maps with a weight $u > 0$ per block. The phase transition occurs at $u = 9/5$, and we have established the existence of three regimes, regarding the size of the largest block, and regarding the scaling limit (and the order of magnitude of distances).

Extension to other models. Our method can be generalised to other models which can be decomposed into appropriate blocks with an underlying tree structure, for example the models described in [BFSS01, Table 3], which is partially reproduced in Table 2. A *triangulation* is a map where all faces have degree 3. It is *irreducible* if every 3-cycle defines a face. In this section, we use the same notation for the various models as in the rest of the article.

Models described in [BFSS01, Table 3] where maps are decomposed into blocks weighted with a weight $u > 0$ undergo a phase transition at the critical value u_C written down in Table 3. More precisely, Theorems 1 to 4 hold for these models with the constants of Table 3. Notice that for the decomposition of general maps into 2-connected maps (*i.e.* the schema linking $\mathcal{M}_1$ and $\mathcal{M}_4$) — which is the case studied in this paper — we get results consistent with Theorem 1. Moreover, the values of u_C and $E(u)$ are consistent since it always holds that $E(u_C) = 1$. Furthermore, for $u = 1$, we retrieve the results of [BFSS01, Table 4]: indeed, our $1 - E(1)$ equals their α_0[6].

Models from [BFSS01, Table 3] are amenable to computations similar to this article's in order to get the values above. We show in Table 3 the most obvious results and models requiring more care will be described in a separate note. In the cases of Table 2, there is $d \in \mathbb{Z}_{>0}$ such that $H(z) = z(1 + M)^d$, and the corresponding law μ^u (except for triangulations) comes naturally as:

$$\mu^u(dm) = \frac{\mathbb{1}_{m\neq 0} u b_m y(u)^m + \mathbb{1}_{m=0}}{uB(y(u)) + 1 - u}, \qquad \mu^u(m) = 0 \quad \text{when} \quad d \nmid m.$$

The cases dealing with triangulations require more care as the series are counted by vertices but the substitution is done on edges in one case, and on internal faces in the other; but keeping this in mind, the same methods can be applied.

For all models, we expect to get similar regimes as in Table 1 (assuming the convergence of the family of blocks is known, as well as diameter estimates). However, conditioning is more difficult for some models, as the size of the map is not always immediately deduced from the size of the Galton Watson tree (*e.g.* for simple triangulations ($\mathcal{T}_2$) decomposed into irreducible triangulations ($\mathcal{T}_3$), the size is the number of leaves of the Galton-Watson tree).

Perspectives. We plan to study similar models in the context of decorated planar maps (*e.g.* tree-rooted maps or Schnyder woods), where the generating series exhibit different singular behaviours. In future work, we also want to investigate more closely the rate of the phase transition at the critical value $u = 9/5$, in analogy to the study of the largest component for the Erdös-Rényi random graph [CFGK20].

Finally, as mentioned in the introduction, the model of maps with a weight u per 2-connected block has been studied as encoding certain discrete spaces of dimension larger than 2, with motivations from theoretical physics [Bon16, Lio17]. The metric properties are however modified via the correspondence, and it would be interesting to determine if the scaling limits remain the same.

[6] It is not obvious at first glance that this should be the case for the case of simple triangulations decomposed into irreducible cores, because each node of the Galton-Watson tree corresponds to a sequence of blocks. However, an extreme condensation phenomenon occurs and the mass is concentrated in only one element of the sequence, so the behaviour remains similar.

maps, $M(z)$	cores, $C(z)$	submaps, $H(z)$
loopless, $M_2(z)$	simple, $M_3(z)$	$z(1+M)$
all, $M_1(z)$	2-connected, $M_4(z)$	$z(1+M)^2$
2-connected $M_4(z)-z$	2-connected simple, $M_5(z)$	$z(1+M)$
bipartite, $B_1(z)$	bipartite simple, $B_2(z)$	$z(1+M)$
bipartite, $B_1(z)$	bipartite 2-connected, $B_4(z)$	$z(1+M)^2$
bipartite 2-connected, $B_4(z)$	bipartite 2-connected simple $B_5(z)$	$z(1+M)$
loopless triangulations, $T_1(z)$	simple triangulations, $z+zT_2(z)$	$z(1+M)^3$
simple triangulations, $T_2(z)$	irreducible triangulations, $T_3(z)$	$z(1+M)^2$

Table 2: Partial reproduction of [BFSS01, Table 3], which describes composition schemas of the form $\mathcal{M}=\mathcal{C}\circ\mathcal{H}$ except the last one where $\mathcal{M}=(1+\mathcal{M})\times\mathcal{C}\circ\mathcal{H}$. The parameter z counts vertices (up to a fixed shift) in the case of triangulations, edges otherwise. Some terms have been changed to correspond to the conventions used in this article.

Maps	Cores	u_C	$E(u)$	$1-E(1)$
$\mathcal{M}_2$	$\mathcal{M}_3$	$\frac{81}{17}$	$\frac{32u}{3(5u+27)}$	$\frac{2}{3}$
$\mathcal{M}_1$	$\mathcal{M}_4$	$\frac{9}{5}$	$\frac{8u}{3(u+3)}$	$\frac{1}{3}$
$\mathcal{M}_4-\mathcal{Z}$	$\mathcal{M}_5$	$\frac{135}{7}$	$\frac{32u}{5(5u+27)}$	$\frac{4}{5}$
$\mathcal{B}_1$	$\mathcal{B}_2$	$\frac{36}{11}$	$\frac{20u}{9(u+4)}$	$\frac{5}{9}$
$\mathcal{B}_1$	$\mathcal{B}_4$	$\frac{52}{27}$	$\frac{40u}{13(u+4)}$	$\frac{5}{13}$
$\mathcal{B}_4$	$\mathcal{B}_5$	$\frac{68}{3}$	$\frac{20u}{17(u+4)}$	$\frac{13}{17}$
$\mathcal{T}_1$	$\mathcal{Z}+\mathcal{Z}\times\mathcal{T}_2$	$\frac{16}{7}$	$\frac{9u}{2(u+8)}$	$\frac{1}{2}$
$\mathcal{T}_2$	$\mathcal{T}_3$	$\frac{64}{37}$	$\frac{27u}{2(32-5u)}$	$\frac{1}{2}$

Table 3: Values of u_C, $E(u)$ when $u\leqslant u_C$ and $1-E(1)$ for all the decomposition schemes of Table 2.

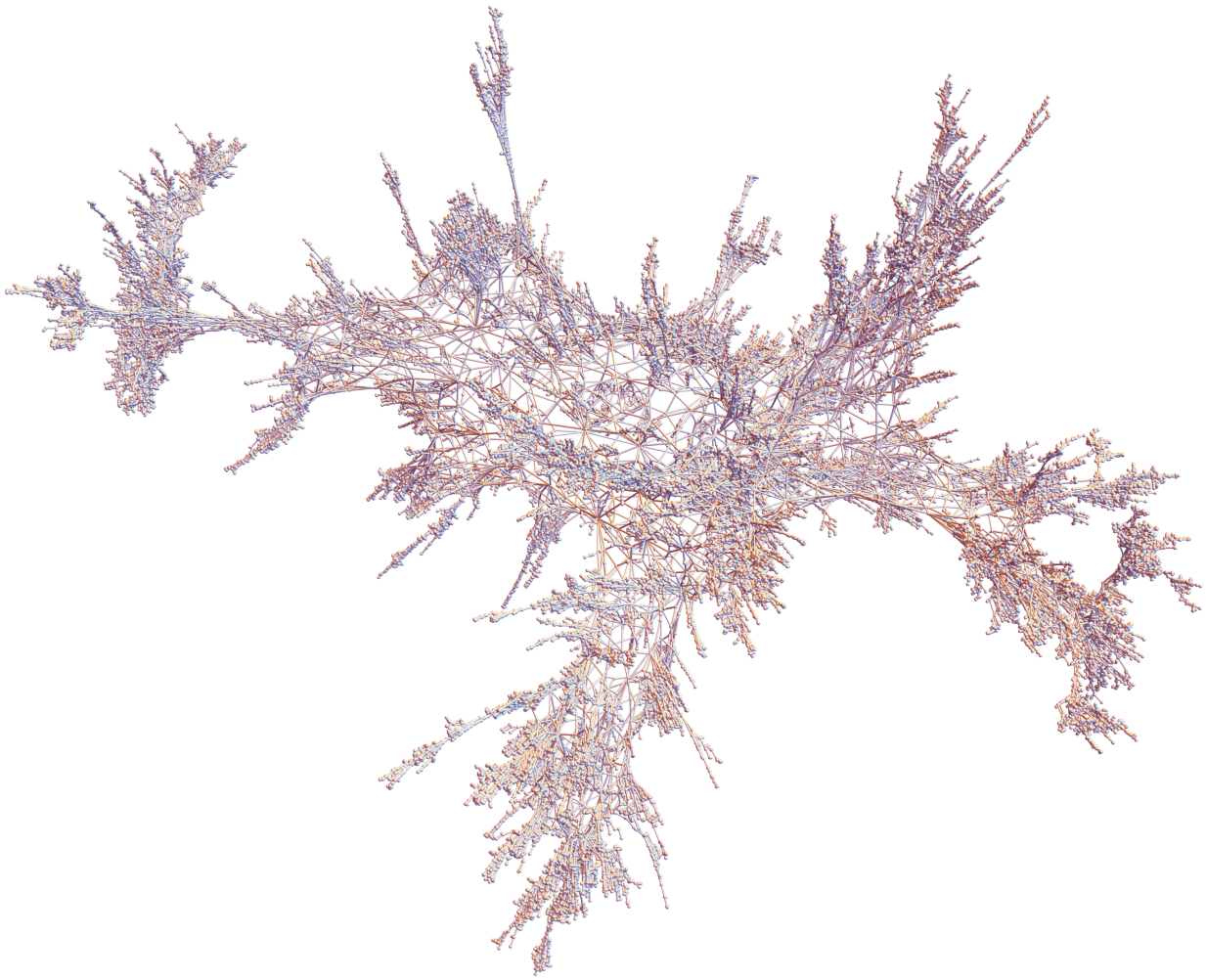

Figure 20: Map drawn according to the subcritical model $\mathbb{P}_{n,1}$ of size around 55 000.

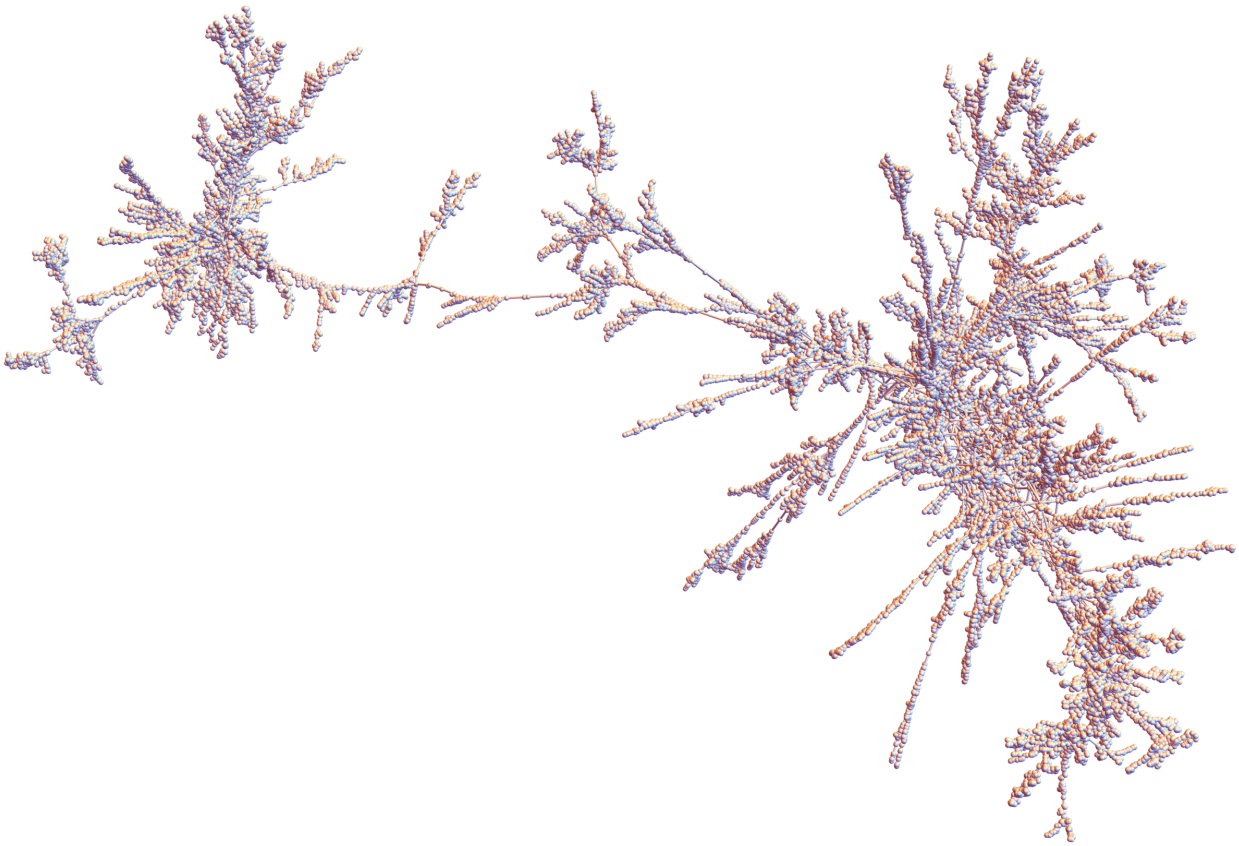

Figure 21: Map drawn according to the subcritical model $\mathbb{P}_{n,8/5}$ of size around 55 000.

Figure 22: Map drawn according to the critical model $\mathbb{P}_{n,9/5}$ of size around 80 000.

Figure 23: Map drawn according to the supercritical model $\mathbb{P}_{n,5/2}$ of size around 75 000.

Figure 24: Map drawn according to the supercritical model $\mathbb{P}_{n,5}$ of size around 50 000.

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