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# On the generalized plane strain assumption for pressurized membranes\*

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We revisit the problem of translation-invariant pressurized membranes that are squeezed without friction between several planes, all parallel to the axis of translation-invariance (such problem involves material and geometric nonlinearities, including contact). Quite remarkably, it was shown by [De Simone and Luongo \(2013\)](#) that such problems simplify considerably under the plane strain assumption. Indeed, the complex initial boundary-value problem reduces to a simple set of non-linear, algebraic equations. We argue that in many practical cases, the plane strain assumption does not hold. Instead, we introduce the *generalized plane strain assumption*, that is necessary to account for the longitudinal equilibrium of the membrane. We show how the equations of [De Simone and Luongo \(2013\)](#) are modified, while remaining extremely simple. We thus define an extended class of problems that become (nearly) tractable analytically.

**Keywords** membranes contact translation-invariance hyperelasticity

## 1. Introduction

Pressurized membranes are in use in various technological domains such as aerospace ([Chandra et al., 2020](#)) or civil engineering ([Chilton, 2013](#)). Their mechanical behavior is well understood ([Khaniki et al., 2023](#); [Firouzi, 2022](#)), allowing for complex numerical simulations ([Fu et al., 2019](#)).

Soft robotics has emerged in the last decade ([Ahmed et al., 2022](#); [Chen and Wang, 2020](#); [Boyras et al., 2018](#)) as an alternative to hard robots that achieve high accuracy but poor adaptivity to their environment ([Walker et al., 2020](#)). According to [Xavier et al. \(2022\)](#), “pneumatic actuation remains the dominant technology in soft robotics due to its low cost and mass, fast response time, and easy implementation.” Such popularity has triggered the need for reliable models of pressurized membranes to control these soft actuators. The models

must account for finite strain hyperelasticity as well as contact, while remaining simple enough to allow for real-time control.

In many instances, the pneumatic actuator can be considered as translation-invariant. More precisely, the following assumptions are made (see [Fig. 2](#)): (1) the initial and current geometries are invariant by translation along the longitudinal axis, (2) the longitudinal dimension of the membrane is large compared to its transverse dimensions, (3) the membrane is subjected to inner pressure. Additionally, the membrane may be in contact with one (or several) longitudinal plane(s) *in the absence of friction or adhesion* (essential assumption).

Translation-invariance leads to considerable mathematical simplifications, since the mechanical state of the membrane is expected to be independent of the longitudinal coordinate (at least, far from the two ends). [De Simone and Luongo \(2013\)](#) further proved two important results (see also [Srivastava and Hui, 2013b,a](#)): first, the cross-section of the deformed membrane is a collection of rectilinear segments (contact zones) and circular arcs (their common radius being imposed by the internal pressure) and second, the hoop stretch and hoop stress resultants are constant along the cross-section. The initial set of non linear partial differential equations therefore reduces to a set of algebraic equations (with two scalar unknowns, namely the hoop stretch and the hoop stress resultant).

These results –upon which most subsequent works rely ([Sachin et al., 2022a,b](#); [Gu et al., 2021](#); [Liu et al., 2021b](#); [Shepherd et al., 2011](#))– were derived under the classical *plane strain assumption* ([De Simone and Luongo, 2013](#); [Srivastava and Hui, 2013b,a](#)), where the longitudinal stretch is assumed to be unity. The membrane is long, but finite and closed by caps at both ends: therefore, the pressurizing gas acts on both ends of the membrane, thus inducing a longitudinal stress and stretch over the *whole* membrane. The convenient plane strain assumption is therefore questionable for membranes.

Existence of longitudinal stresses is well-known in pressure vessel engineering: in a cylindrical pressure vessel, the longitudinal stress resultant is equal to half the transverse stress resultant. Another well-known example is the twisting balloon: during inflation, its total length increases and the longitudinal stretch is greater than one (closely related is the bulging instability [Kyriakides and Yu-Chung, 1990](#); [Lestringant and Audoly, 2018](#)).

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The goal of this paper is to question the plane strain assumption for translation-invariant membranes. Observing that translation-invariant membranes fit the following description (Zhenye and Shiping, 1990): “the deformation situation of any two arbitrary cross sections perpendicular to the longitudinal axis are identical, while the strain and displacement components of various points of the same plane may be different”, we argue in this paper that such systems should really be analyzed under the *generalized plane strain* assumption. We therefore allow for a constant ( $\neq 1$ ) longitudinal stretch. Quite remarkably, we then show that most of the theoretical results obtained by De Simone and Luongo (2013) under the plane strain assumption remain valid under the generalized plane strain assumption. This is the principal outcome of this paper, from which it results that simulation of a translation-invariant, pressurized membrane can still be performed very efficiently (there are now *four* scalar unknowns, rather than two: two stretches and two stress resultants). We also show on a iconic example that the plane strain assumption remains acceptable so long as the hoop stretch remains close to unity. For larger stretches, this assumption leads to significant errors.

The paper is organized as follows: the nonlinear theory of membranes is summarized in Sec. 2. The general membrane equations are then specialized to translation-invariant membranes in Sec. 3, where it is shown that under the generalized plane strain assumption, the resulting problem simplifies considerably. An example is considered in Sec. 4, where the plane strain and generalized plane strain assumptions are compared.

## 2. Background

The present section provides a brief overview of the theory of hyperelastic membranes (see also Gurtin and Murdoch, 1975; Haughton and Ogden, 1978a,b; Erbay, 1997; Steigmann, 2009, for more details). No assumptions are made at this point on the geometry: the theory will be specialized to translation-invariant membranes in Sec. 3.

Note that as much as possible, we stick in this paper to the convention that capital letters refer to the initial configuration, while small letters refer to the current (deformed) configuration. Besides, Greek indices ( $\alpha, \beta, \gamma, \dots$ ) span the  $\{1, 2\}$  index set, while latin indices ( $i, j, k, \dots$ ) span the  $\{1, 2, 3\}$  index set.

### 2.1. Geometry

The initial configuration is defined through the mapping  $(\Xi^1, \Xi^2) \mapsto \mathbf{X}(\Xi^1, \Xi^2) \in \mathbb{R}^3$ , where  $\Xi^1$  and  $\Xi^2$  are the curvilinear coordinates and  $\mathbf{X}$  is the current point on the reference surface of the membrane. Similarly, the deformed configuration is defined through the mapping  $(\xi^1, \xi^2) \mapsto \mathbf{x}(\xi^1, \xi^2) \in \mathbb{R}^3$  (see Fig. 1). The corresponding covariant and contravariant bases are defined as follows

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{X}}{\partial \Xi^\alpha}, \quad \mathbf{a}_\alpha = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} \quad \text{and} \quad \mathbf{A}^\beta \cdot \mathbf{A}_\alpha = \mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta_\alpha^\beta. \quad (1)$$

We further introduce the unit normal  $\mathbf{A}_3$  (resp.  $\mathbf{a}_3$ ) and curvature tensor  $\mathbf{B}$  (resp.  $\mathbf{b}$ ) of the initial (resp. current) configura-

tion

$$\begin{aligned} \mathbf{A}_3 &= \frac{\mathbf{A}_1 \times \mathbf{A}_2}{\|\mathbf{A}_1 \times \mathbf{A}_2\|}, & \mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}, \\ \mathbf{B} &= -\frac{\partial \mathbf{A}_3}{\partial \Xi^\alpha} \otimes \mathbf{A}^\alpha, & \mathbf{b} &= -\frac{\partial \mathbf{a}_3}{\partial \xi^\alpha} \otimes \mathbf{a}^\alpha. \end{aligned} \quad (2)$$

### 2.2. Transformation

The transformation is defined by the mapping  $(\Xi^1, \Xi^2) \mapsto (\xi^1, \xi^2) = \Phi(\Xi^1, \Xi^2)$  that maps the initial position  $\mathbf{X}(\Xi^1, \Xi^2)$  (material coordinates) onto the current position  $\mathbf{x}(\xi^1, \xi^2)$  (spatial coordinates). The gradient of the transformation,  $\mathbf{F}$ , maps the small material vector  $d\mathbf{X}$  onto its spatial representation  $d\mathbf{x}$ :  $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$ . We have, from Eq. (1):

$$\mathbf{A}^\alpha \cdot d\mathbf{X} = \mathbf{A}^\alpha \cdot \frac{\partial \mathbf{X}}{\partial \Xi^\beta} d\Xi^\beta = \mathbf{A}^\alpha \cdot \mathbf{A}_\beta d\Xi^\beta = \delta_\beta^\alpha d\Xi^\beta = d\Xi^\alpha \quad (4)$$

therefore, using the chain rule

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^\alpha} \frac{\partial \xi^\alpha}{\partial \Xi^\beta} d\Xi^\beta = \frac{\partial \xi^\alpha}{\partial \Xi^\beta} (\mathbf{a}_\alpha \otimes \mathbf{A}^\beta) \cdot d\mathbf{X} \quad (5)$$

and

$$\mathbf{F} = \frac{\partial \xi^\alpha}{\partial \Xi^\beta} \mathbf{a}_\alpha \otimes \mathbf{A}^\beta. \quad (6)$$

Note that the tensor introduced above lives in part in the tangent plane to the initial configuration ( $\mathbf{A}_1, \mathbf{A}_2$ ) and in part in the tangent plane to the current configuration ( $\mathbf{a}_1, \mathbf{a}_2$ ), since

$$\mathbf{F} \cdot \mathbf{A}_3 = \mathbf{0} \quad \text{and} \quad \mathbf{a}_3 \cdot \mathbf{F} = \mathbf{0}. \quad (7)$$

To close this section on the transformation of the membrane, we introduce the (2d) right Cauchy-Green tensor,  $\mathbf{C} = \mathbf{F}^\top \cdot \mathbf{F}$ . This tensor is symmetric and lives in the tangent plane to the initial configuration

$$\mathbf{C}^\top = \mathbf{C}, \quad \mathbf{C} \cdot \mathbf{A}_3 = \mathbf{0} \quad \text{and} \quad \mathbf{A}_3 \cdot \mathbf{C} = \mathbf{0}. \quad (8)$$

This tensor will be used to express the constitutive law of the membrane.

### 2.3. Equilibrium

The internal state of stress of the membrane is fully defined by the stress resultants  $\mathbf{N}$  (membrane forces). The second-order, symmetric tensor  $\mathbf{N}$  is a *spatial* tensor that has all components in the tangent plane to the *deformed* configuration

$$\mathbf{N}^\top = \mathbf{N}, \quad \mathbf{N} \cdot \mathbf{a}_3 = \mathbf{0} \quad \text{and} \quad \mathbf{a}_3 \cdot \mathbf{N} = \mathbf{0}. \quad (9)$$

Equilibrium of the membrane is governed by the following general equation

$$\frac{\partial \mathbf{N}}{\partial \xi^\alpha} \cdot \mathbf{a}^\alpha + \mathbf{f} = \mathbf{0}, \quad (10)$$

which expresses that the applied resultant forces  $\mathbf{f}$  (surface density, spatial description) are equilibrated by the *divergence* of  $\mathbf{N}$ . Note that the above equation is written on the *current*

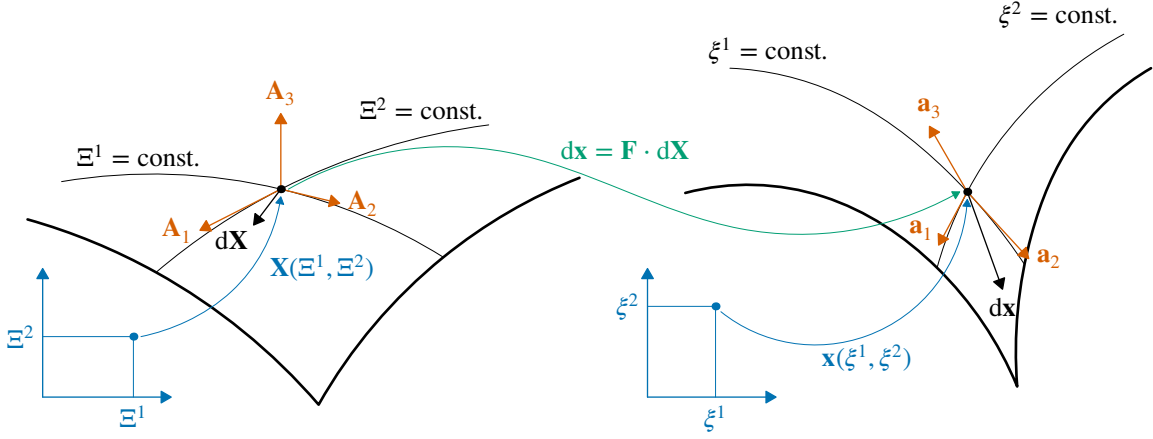


Figure 1: The initial (left) and current (right) configurations  $\Sigma$  and  $\zeta$  of the membrane. The figure represents the mappings between the parameter spaces  $(\Xi^1, \Xi^2)$ ,  $(\xi^1, \xi^2)$  and the surfaces  $\Sigma$ ,  $\zeta$ , as well as the gradient of the transformation.

(deformed) configuration. Projection along the normal delivers the classical algebraic equation ( $f_3 = \mathbf{f} \cdot \mathbf{a}_3$ , see proof in Appendix A)

$$\mathbf{b} : \mathbf{N} + f_3 = 0. \quad (11)$$

Besides the field equations (10) and (11), the following continuity condition must hold at any point  $\mathbf{x}$  of the membrane and for any tangent vector  $\mathbf{v}$  to the membrane at  $\mathbf{x}$

$$[\mathbf{N}(\mathbf{x} + \epsilon \mathbf{v}) - \mathbf{N}(\mathbf{x} - \epsilon \mathbf{v})] \cdot \mathbf{v} \rightarrow 0 \quad \text{when} \quad \epsilon \rightarrow 0. \quad (12)$$

## 2.4. Constitutive equations

The constitutive material is isotropic, incompressible and hyperelastic. We introduce the principal stretches  $\lambda_I$ ,  $\lambda_{II}$  and  $\lambda_{III}$  associated with the (3d) right Cauchy–Green tensor  $\mathbf{C}$ . The (3d) strain energy density is expressed as a function of the principal stretches

$$\text{Strain energy density} = \mu \bar{W}(\lambda_I, \lambda_{II}, \lambda_{III}), \quad (13)$$

where the dimensionless function  $\bar{W}$  is scaled appropriately in order to ensure that  $\mu$  is the shear modulus of the material. The constitutive equations then read

$$\sigma_K = \mu \lambda_K \frac{\partial \bar{W}}{\partial \lambda_K} + \eta, \quad K \in \{I, II, III\} \quad (14)$$

(no summation on  $K$ ). In the above equations,  $\sigma_I$ ,  $\sigma_{II}$  and  $\sigma_{III}$  are the principal Cauchy stresses and  $\eta$  is the Lagrange multiplier associated with the incompressibility condition.

For thin membranes, following Haughton and Ogden (1978a), we introduce the *two* principal stretches  $\lambda_I$  and  $\lambda_{II}$  associated with the (2d) right Cauchy–Green tensor  $\mathbf{C}$  introduced in Sec. 2.2 and the reduced potential

$$W(\lambda_I, \lambda_{II}) = \bar{W}(\lambda_I, \lambda_{II}, \lambda_{III}) \quad \text{with} \quad \lambda_{III} = (\lambda_I \lambda_{II})^{-1}, \quad (15)$$

where the third principal stretch is defined so as to enforce the incompressibility constraint. Then, the constitutive equations of the membrane read, in principal components

$$N_I = \lambda_I \mu h \frac{\partial W}{\partial \lambda_I} \quad \text{and} \quad N_{II} = \lambda_{II} \mu h \frac{\partial W}{\partial \lambda_{II}}, \quad (16)$$

where  $h$  denotes the thickness of the deformed membrane, while  $N_I$  and  $N_{II}$  are the principal values of  $\mathbf{N}$ . Since the normal to the membrane coincides with the direction of the third principal stretch, we have

$$h = \lambda_{III} H = \frac{H}{\lambda_I \lambda_{II}}, \quad (17)$$

where  $H$  is the thickness of the underformed membrane. Combining Eqs. (16) and (17), we find the constitutive relations

$$N_I = \frac{\mu H}{\lambda_{II}} \frac{\partial W}{\partial \lambda_I} \quad \text{and} \quad N_{II} = \frac{\mu H}{\lambda_I} \frac{\partial W}{\partial \lambda_{II}}. \quad (18)$$

Note that these constitutive equations are expressed on the current (deformed) configuration: the stress-resultants  $N_I$  and  $N_{II}$  are the membrane equivalents of the principal Cauchy stresses. Observe also that, owing to the fact that  $\lambda_{III} = \lambda_I^{-1} \lambda_{II}^{-1}$  is no longer a free variable, no Lagrange multiplier is necessary in Eq. (18) to ensure incompressibility.

In the remainder of this paper, we consider the family of generalized neo-Hookean hyperelastic potentials recently introduced by Anssari-Benam and Bucchi (2021). These potentials are derived from the statistical analysis of a network of freely jointed molecular chains. They are defined by only two material constants (the shear modulus  $\mu$  and the number  $\mathcal{N}$  of Kuhn segments of a chain)

$$\bar{W}(\lambda_I, \lambda_{II}, \lambda_{III}) = \mathcal{N} \frac{3 - 3\mathcal{N}}{1 - 3\mathcal{N}} \left( \frac{I_1 - 3}{6\mathcal{N}} - \ln \frac{I_1 - 3\mathcal{N}}{3 - 3\mathcal{N}} \right), \quad (19)$$

with

$$\lambda_I \lambda_{II} \lambda_{III} = 1 \quad \text{and} \quad I_1 = \lambda_I^2 + \lambda_{II}^2 + \lambda_{III}^2. \quad (20)$$

These generalized neo-Hookean materials were found to account accurately for the inflation of spherical and cylindrical membranes (Anssari-Benam et al., 2022), including limit-point and inflation-jump instabilities. Note that in the papers by Anssari-Benam and Bucchi (2021) and Anssari-Benam et al. (2022), the symbol  $\mu$  denotes a convenient constant; it differs from the shear modulus that is denoted  $\mu_0$ . In the

present paper, we do not introduce the convenient constant and use the shear modulus exclusively. Plugging  $\lambda_{III} = \lambda_I^{-1} \lambda_{II}^{-1}$  delivers the following reduced potential

$$W(\lambda_I, \lambda_{II}) = \mathcal{N} \frac{3 - 3\mathcal{N}}{1 - 3\mathcal{N}} \left( \frac{\lambda_I^2 + \lambda_{II}^2 + \lambda_I^{-2} \lambda_{II}^{-2} - 3}{6\mathcal{N}} - \ln \frac{\lambda_I^2 + \lambda_{II}^2 + \lambda_I^{-2} \lambda_{II}^{-2} - 3\mathcal{N}}{3 - 3\mathcal{N}} \right), \quad (21)$$

Upon substitution into Eqs. (18), the following constitutive equations are found

$$\frac{N_I}{\mu H} = n(\lambda_I^4 \lambda_{II}^2 - 1) \quad \text{and} \quad \frac{N_{II}}{\mu H} = n(\lambda_I^2 \lambda_{II}^4 - 1), \quad (22)$$

with

$$n = \frac{(1 - \mathcal{N}) [1 + \lambda_I^2 \lambda_{II}^2 (\lambda_I^2 + \lambda_{II}^2 - 9\mathcal{N})]}{\lambda_I^3 \lambda_{II}^3 (1 - 3\mathcal{N}) [1 + \lambda_I^2 \lambda_{II}^2 (\lambda_I^2 + \lambda_{II}^2 - 3\mathcal{N})]}. \quad (23)$$

In the applications discussed in Sec. 4, we selected  $\mathcal{N} = 30$  which is typical of rubber (Anssari-Benam et al., 2022). The constitutive equations of membranes are also derived in Appendix B for Mooney–Rivlin materials.

### 3. Translation-invariant membranes

The present section is the central part of this paper. It specializes to translation-invariant membranes the general theory exposed in Sec. 2. It is recalled that a membrane is translation invariant if, *far from its two ends*, its geometry, deformations and state of stress are all independent of the longitudinal coordinate. The geometry of the membrane is then fully defined by its cross-section, which reduces to a closed curve (see Fig. 2).

Sec. 3 is organized as follows. We first define the geometry of the membrane in Sec. 3.1. We then discuss local equilibrium in Sec. 3.2. As this local analysis delivers no information on the longitudinal membrane stress resultant, it is complemented in Sec. 3.3 by a global analysis of one half of the membrane, cut along the  $z = 0$  plane. In order to account for deformations of the membrane, the generalized plane strain assumption is introduced in Sec. 3.4, where the transformation is defined. The constitutive relations of the membrane are then introduced in Sec. 3.5. The resulting closed system of algebraic equations is summarized in Sec. 3.6.

#### 3.1. Geometry of translation-invariant membranes

The cross-section of the membrane in the initial (*resp.* deformed) configuration is the plane curve  $\Gamma$  (*resp.*  $\gamma$ ). Note that both  $\Gamma$  and  $\gamma$  are *simple, closed* curves;  $\mathcal{L}_\Gamma$  and  $\mathcal{L}_\gamma$  denote their respective length.  $\Sigma$  and  $\zeta$  denote the regions bounded by  $\Gamma$  and  $\gamma$ , respectively;  $\mathcal{A}_\Sigma$  and  $\mathcal{A}_\zeta$  denote their surface area.

These curves are parametrized by the arc-length  $S$  (*resp.*  $s$ ). The current points on  $\Gamma$  and  $\gamma$  are denoted  $\bar{\mathbf{X}}(S) = \bar{X}(S) \mathbf{e}_x + \bar{Y}(S) \mathbf{e}_y$  and  $\bar{\mathbf{x}}(s) = \bar{x}(s) \mathbf{e}_x + \bar{y}(s) \mathbf{e}_y$ , which are both radius-vectors in the  $(x, y)$  plane.

Introducing the unit tangents  $\mathbf{T}$  and  $\mathbf{t}$ , the *inner* normals  $\mathbf{N} = \mathbf{e}_z \times \mathbf{T}$  and  $\mathbf{n} = \mathbf{e}_z \times \mathbf{t}$ , and the radii of curvature  $R$  and  $r$ , we have

$$\mathbf{T} = \frac{d\mathbf{X}}{dS}, \quad \mathbf{t} = \frac{d\mathbf{x}}{ds}, \quad \frac{d\mathbf{T}}{dS} = \frac{\mathbf{N}}{R} \quad \text{and} \quad \frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{r} \quad (24)$$

and it is further shown in Appendix C that

$$\oint_\gamma \mathbf{t} ds = \mathbf{0} \quad \text{and} \quad \oint_\gamma \bar{\mathbf{x}} \times \mathbf{t} ds = 2\mathcal{A}_\zeta \mathbf{e}_z, \quad (25)$$

where the “ $\oint$ ” symbol emphasizes the fact that  $\gamma$  is a closed contour.

Any point  $\mathbf{X}$  (*resp.*  $\mathbf{x}$ ) of the membrane on the initial (*resp.* current) configuration is then parametrized by the curvilinear coordinates  $S$  and  $Z$  (*resp.*  $s$  and  $z$ ), such that

$$\mathbf{X}(S, Z) = \bar{\mathbf{X}}(S) + Z \mathbf{e}_z \quad \text{and} \quad \mathbf{x}(s, z) = \bar{\mathbf{x}}(s) + z \mathbf{e}_z. \quad (26)$$

The covariant basis is orthonormal and coincides with the contravariant basis

$$\mathbf{A}_S = \mathbf{A}^S = \frac{\partial \mathbf{X}}{\partial S} = \frac{d\bar{\mathbf{X}}}{dS} = \mathbf{T}, \quad \mathbf{A}_Z = \mathbf{A}^Z = \frac{\partial \mathbf{X}}{\partial Z} = \mathbf{e}_z, \quad (27)$$

$$\mathbf{a}_s = \mathbf{a}^s = \frac{\partial \mathbf{x}}{\partial s} = \frac{d\bar{\mathbf{x}}}{ds} = \mathbf{t}, \quad \mathbf{a}_z = \mathbf{a}^z = \frac{\partial \mathbf{x}}{\partial z} = \mathbf{e}_z, \quad (28)$$

while the normals are such that

$$\mathbf{A}_3 = \frac{\mathbf{A}_z \times \mathbf{A}_S}{\|\mathbf{A}_z \times \mathbf{A}_S\|} = \frac{\mathbf{e}_z \times \mathbf{T}}{\|\mathbf{e}_z \times \mathbf{T}\|} = \mathbf{N}, \quad (29)$$

$$\mathbf{a}_3 = \frac{\mathbf{a}_z \times \mathbf{a}_s}{\|\mathbf{a}_z \times \mathbf{a}_s\|} = \frac{\mathbf{e}_z \times \mathbf{t}}{\|\mathbf{e}_z \times \mathbf{t}\|} = \mathbf{n}. \quad (30)$$

Note that with the adopted orientation of the normal,  $r > 0$  in the cases considered here, and the curvature tensor reads:  $\mathbf{b} = \mathbf{t} \otimes \mathbf{t}/r$  (see Fig. 2).

#### 3.2. Local equilibrium of translation-invariant membranes

De Simone and Luongo (2013) observed that equilibrium of translation-invariant membranes severely constraints their shape. However, their proof was based on the restrictive assumption that the state of stress of the membrane was fully defined by only *one* so-called tension. Then, in their own words, their “problem formally coincides with that which governs planar cables”. Through elementary equilibrium considerations, they show that the tension is *constant* in the membrane. As a consequence, for a wide class of materials, the corresponding stretch is *also constant*. In other words, the state of the membrane is fully defined by only *two scalars*: the tension and the stretch. Besides, the initial boundary-value problem that governs the equilibrium of the membrane now reduces to a set of *algebraic* equations.

As discussed in Sec. 2, the state of stress of the most general membrane is in fact defined by *three* (rather than one) membrane stress resultants, and the proof of De Simone and Luongo (2013) must be revisited. In what follows, we prove that their main conclusion remains essentially unchanged: “when no outside forces act on the membrane, the original

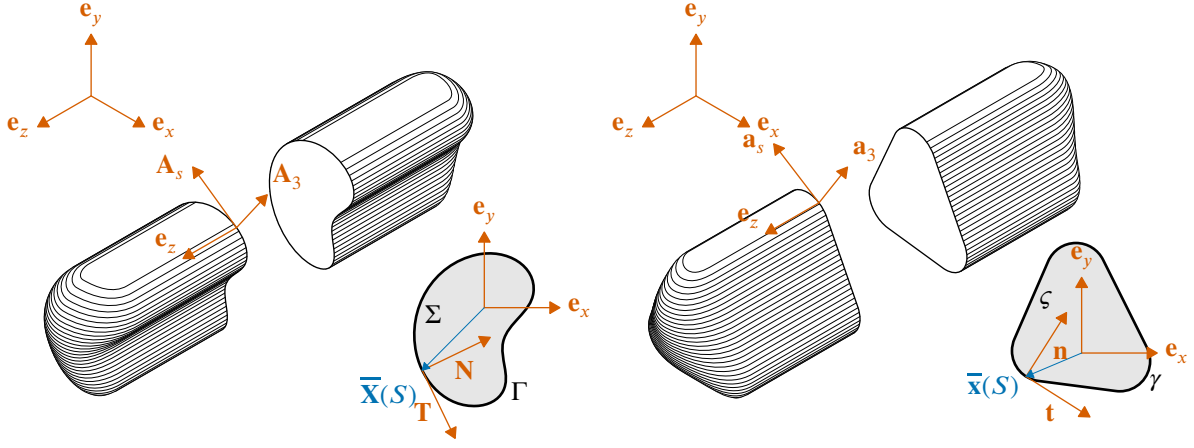


Figure 2: The geometry of a translation-invariant membrane is fully defined by its cross-section (a closed, simple curve). The initial cross-section (left) is defined by the contour  $\Gamma$ ; its interior is the domain  $\Sigma$ . Likewise, the deformed cross-section is defined by the contour  $\gamma$ ; its interior is the domain  $\zeta$ . As shown in Sec 3.2,  $\gamma$  is a collection of rectilinear segments and circular arcs of constant radius. Note that the initial cross-section  $\Gamma$  is somewhat fictitious, as the membrane cannot be deployed under null overpressure; what really matters is the initial *length*  $\mathcal{L}_\Gamma$  of the cross-section. The radius-vector of the current point on the initial and deformed configurations are  $\bar{\mathbf{X}}(S)$  and  $\bar{\mathbf{x}}(s)$ , respectively.

infinite-dimensional problem can be formulated as a much simpler finite-dimensional problem [...] If the material is visco-elastic, the problem is mixed algebraic-differential in time; if it is elastic, it further degenerates into algebraic.” The proof is however now more involved, and the number of algebraic unknowns increases.

We first decompose the tensor of membrane stress resultants  $\mathbf{N}$  in the  $(\mathbf{t}, \mathbf{e}_z)$  orthonormal basis

$$\mathbf{N} = N_s \mathbf{t} \otimes \mathbf{t} + N_{sz} \mathbf{e}_z \otimes \mathbf{e}_z + N_{sz} (\mathbf{t} \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{t}). \quad (31)$$

Using the general equilibrium equation (10) with  $\partial_z \mathbf{N} = \mathbf{0}$ ,  $\partial_s \mathbf{t} = r^{-1} \mathbf{a}_3$  and  $\mathbf{N} \cdot \mathbf{a}_3 = \mathbf{0}$ , we find that

$$\begin{aligned} \mathbf{0} &= \partial_s \mathbf{N} \cdot \mathbf{t} + \mathbf{p} = \partial_s (\mathbf{N} \cdot \mathbf{t}) - \mathbf{N} \cdot \partial_s \mathbf{t} + \mathbf{f} \\ &= \partial_s (N_s \mathbf{t} + N_{sz} \mathbf{e}_z) - \frac{\mathbf{N} \cdot \mathbf{a}_3}{r} + \mathbf{f} \\ &= \partial_s N_s \mathbf{t} + \partial_s N_{sz} \mathbf{e}_z + \frac{N_s}{r} \mathbf{a}_3 + \mathbf{f}. \end{aligned} \quad (32)$$

It is assumed that the applied resultant forces have no out-of-plane component ( $\mathbf{f} \cdot \mathbf{e}_z = 0$ ). Therefore  $N_{sz}$  is constant along  $\gamma$  and

$$\partial_s N_s + \mathbf{f} \cdot \mathbf{t} = 0 \quad \text{and} \quad \frac{N_s}{r} + \mathbf{f} \cdot \mathbf{a}_3 = 0. \quad (33)$$

Two particular cases of pressurized membranes are discussed in the remainder of this section.

**Free section of a pressurized membrane** We consider a section of the pressurized membrane that is free to expand. The loading is such that  $\mathbf{f} = -p \mathbf{a}_3$ , where  $p$  is the uniform pressure of the enclosed gas (note that  $\mathbf{a}_3$  is the *inner* normal), Eq. (33) delivers

$$\partial_s N_s = 0 \quad \text{and} \quad N_s = p r. \quad (34)$$

In a free section of a pressurized membrane, both the membrane stresses  $N_s$  and  $N_{sz}$  are therefore constant. From the last equation, we further deduce that the radius of curvature  $r$  is also constant. In other words, the membrane is a segment of circular cylinder with radius  $r = N_s/p$ .

**Section of a pressurized membrane in frictionless contact with a plane** We consider a section of the membrane that is constrained by contact with a plane. Frictionless contact is frequently assumed in the literature (Tamadapu and Dasgupta, 2014; Kumar et al., 2021; Liu et al., 2021a; Yang et al., 2021; Khaniki et al., 2023) and we will also adopt this assumption in the remainder of this paper.

In the constrained section of the membrane,  $r \rightarrow +\infty$  and it results from Eq. (33) that  $\mathbf{f} \cdot \mathbf{a}_3 = 0$ . In other words, the out-of-plane component of the surface forces vanish, as the internal pressure is fully balanced by the reaction forces exerted by the plane support. Besides, under the no friction assumption, the in-plane components of  $\mathbf{f}$  must also vanish. Therefore,  $\mathbf{f} = \mathbf{0}$  and it is again found that  $N_s$  and  $N_{sz}$  are constant along the section of  $\gamma$  under consideration.

It results from the above discussion that the shape of a translation-invariant, pressurized membrane is fully defined. In unconstrained segments, the membrane takes a cylindrical shape and  $N_s$  and  $N_{sz}$  are constant throughout the segment; besides, the radius of curvature  $r$  is given by  $N_s = p r$ . In the segments that are constrained by frictionless, plane contact conditions, the shape of the membrane is prescribed (rectilinear) and  $N_s$  and  $N_{sz}$  are also constant throughout the segment. In turn, the continuity conditions (12) require (for  $\mathbf{v} = \mathbf{t}$ ) that  $N_s$  and  $N_{sz}$  be continuous at the interface between two segments; in other words,  $N_s$  and  $N_{sz}$  are *constant in the whole cross-section*. The membrane stresses  $N_s$  and  $N_{sz}$  are therefore *scalar unknowns*. These conclusions are similar to those of De Simone and Luongo (2013), but for the fact that these

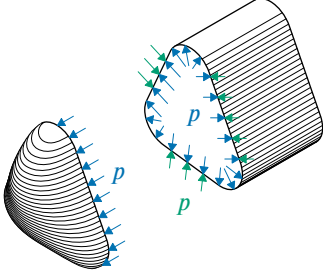


Figure 3: The subsystem under consideration in Sec. 3.3. The cap (left part) is subjected to pressure forces that are equivalent to pressure forces applied to the plane surface that closes the cap. The lateral surface is subjected to pressure forces and, contact forces along the rectilinear segments of the cross-section.

authors disregarded  $N_{sz}$  and  $N_z$ .

**Remark 3.1.** Note that for adhesive contact or contact with friction, neither  $N_s$ , nor  $N_{sz}$  are constant in the rectilinear segments of the membrane. Indeed, the surface forces now have a tangential component  $\mathbf{f} \cdot \mathbf{t} \neq 0$  and/or  $\mathbf{f} \cdot \mathbf{e}_z \neq 0$ . As a consequence,  $N_s$  and  $N_{sz}$  may vary according to the equilibrium equation (32) in the contact areas:  $\partial_s N_s + \mathbf{f} \cdot \mathbf{t} = 0$  and  $\partial_s N_{sz} + \mathbf{f} \cdot \mathbf{e}_z = 0$ . However, the continuity condition (12) prescribes the value of  $N_s$  and  $N_{sz}$  at both ends of the rectilinear segment. In particular,  $N_s = pr$  in all unconstrained segments of the membrane; therefore  $N_s = pr$  at both ends of any rectilinear segment and the transverse resultant of the friction/adhesion forces is null.

In the longitudinal direction, the situation is less clear: it is doubtful that useful, generic, properties of  $N_{sz}$  can emerge from equilibrium considerations only: both the constitutive law of the membrane and –more importantly– the nature of the contact itself must be specified.

### 3.3. Global equilibrium of pressurized membranes

At this point, the local equilibrium equations have delivered no condition on the longitudinal stress resultant  $N_z$ . Recalling that the membrane is closed at both ends, we consider the equilibrium of the subsystem  $z \geq z_0$ , where  $z_0$  is arbitrary. Expressing that the total resultant force and moment exerted on this subsystem must vanish in order to ensure equilibrium delivers some additional conditions on  $N_{sz}$  and  $N_z$ .

We first consider the resultant of *external* forces:  $\mathbf{R}_{\text{ext}} = \mathbf{R}_{\text{cap}} + \mathbf{R}_{\text{lat}}$ , where  $\mathbf{R}_{\text{cap}}$  applies to the membrane's cap, while  $\mathbf{R}_{\text{lat}}$  apply to its lateral surface (see Fig. 3). The membrane's cap is subjected to pressure forces only, the resultant of which is readily evaluated in Appendix D:  $\mathbf{R}_{\text{cap}} = p \mathcal{A}_\zeta \mathbf{e}_z$ .

The lateral surface is subjected to both pressure *and* contact forces; quite remarkably, their resultant vanishes. Indeed, it has been shown in Sec. 3.2 that where contact occur, the resultant forces vanish:  $\mathbf{f} = \mathbf{0}$ . In other words, contact forces balance pressure forces exactly along the rectilinear segments of the cross-section. Where the membrane is unconstrained, it is subjected to pressure forces only (no contact forces) and

its cross-section takes the shape of a circular arc. It is shown in Appendix E that all these arcs add up to form a full *closed* circle: the resultant of pressure forces therefore vanishes. To sum up, the resultant of pressure and contact forces vanish both in the rectilinear and in the circular segments of the membrane. Therefore,  $\mathbf{R}_{\text{lat}} = \mathbf{0}$ .

Gathering the above results delivers  $\mathbf{R}_{\text{ext}} = p \mathcal{A}_\zeta \mathbf{e}_z$ .

Next, we consider the resultant *internal* forces  $\mathbf{R}_{\text{int}}$ . Internal forces are exerted along the border  $z = z_0$  of the subsystem under consideration and their linear density (per unit length measured on the current configuration) is, by definition of the membrane stress resultants,  $\mathbf{N} \cdot (-\mathbf{e}_z) = -N_z \mathbf{e}_z - N_{sz} \mathbf{t}$ . Therefore

$$\mathbf{R}_{\text{int}} = - \oint_\gamma (N_z \mathbf{e}_z + N_{sz} \mathbf{t}) ds. \quad (35)$$

Recalling that  $N_{sz} = \text{const}$ , it results from Eq. (25) that

$$\mathbf{R}_{\text{int}} = - \oint_\gamma N_z ds \mathbf{e}_z. \quad (36)$$

Equilibrium of the subsystem under consideration requires that  $\mathbf{R}_{\text{int}} + \mathbf{R}_{\text{ext}} = \mathbf{0}$ , from which it results that

$$\oint_\gamma N_z ds = p \mathcal{A}_\zeta. \quad (37)$$

We now consider the balance of moments along the  $\mathbf{e}_z$  axis to show that  $N_{sz} = 0$ . All moments being evaluated with respect to the (arbitrary) origin, we have with obvious notations  $\mathbf{M}_{\text{int}} + \mathbf{M}_{\text{ext}} = \mathbf{0}$ . In the present derivation, we are concerned with the longitudinal components of these moments only,  $\mathbf{e}_z \cdot \mathbf{M}_{\text{int}}$  and  $\mathbf{e}_z \cdot \mathbf{M}_{\text{ext}}$ .

External forces apply to the membrane's cap and its lateral surface:  $\mathbf{M}_{\text{ext}} = \mathbf{M}_{\text{cap}} + \mathbf{M}_{\text{lat}}$ . From Appendix D, we have  $\mathbf{M}_{\text{cap}} = p \mathcal{A}_\zeta \mathbf{g}_\zeta \times \mathbf{e}_z$ , where  $\mathbf{g}_\zeta$  is the radius-vector of the center of gravity of  $\zeta$ . We therefore have  $\mathbf{e}_z \cdot \mathbf{M}_{\text{cap}} = 0$ . In Appendix E, it is further shown that

$$\mathbf{e}_z \cdot \mathbf{M}_{\text{lat}} = -pL \int_{\tilde{\gamma}} \bar{\mathbf{x}} \cdot \mathbf{t} ds, \quad (38)$$

where  $L$  denotes the length (in the  $z$  direction) of the lateral surface of the subsystem, and  $\tilde{\gamma}$  is the union of the circular arcs that compose  $\gamma$ . Finally,

$$\begin{aligned} \mathbf{M}_{\text{int}} &= - \oint_\gamma \mathbf{x} \times (N_z \mathbf{e}_z + N_{sz} \mathbf{t}) ds \\ &= - \oint_\gamma (z \mathbf{e}_z + \bar{\mathbf{x}}) \times (N_z \mathbf{e}_z + N_{sz} \mathbf{t}) ds \end{aligned} \quad (39)$$

and (since  $N_{sz} = \text{const}$ )

$$\begin{aligned} \mathbf{e}_z \cdot \mathbf{M}_{\text{int}} &= -N_{sz} \oint_\gamma \mathbf{e}_z \cdot (\bar{\mathbf{x}} \times \mathbf{t}) ds = -N_{sz} \mathbf{e}_z \cdot \oint_\gamma \bar{\mathbf{x}} \times \mathbf{t} ds \\ &= -2\mathcal{A}_\zeta N_{sz}, \end{aligned} \quad (40)$$

where we have used Eq. (25)<sub>2</sub>. Equilibrium of the subsystem therefore requires

$$2\mathcal{A}_\zeta N_{sz} + pL \int_{\tilde{\gamma}} \bar{\mathbf{x}} \cdot \mathbf{t} ds = 0. \quad (41)$$

The first term is independent of the (arbitrary) length  $L$  of the subsystem. Therefore, we must have

$$N_{sz} = 0 \quad \text{and} \quad \int_{\bar{\gamma}} \bar{\mathbf{x}} \cdot \mathbf{t} \, ds = 0. \quad (42)$$

The last condition provides additional constraints to the deformed geometry of the membrane. In the situations considered in this paper, this condition is automatically satisfied owing to symmetries. It will therefore be discarded below. To sum up, it has been shown in Secs. 3.2 and 3.3 that

$$N_s = \text{const}, \quad N_{sz} = 0, \quad \text{and} \quad \oint_{\gamma} N_z \, ds = p \mathcal{A}_s. \quad (43)$$

It is remarkable that Eq. (43) results from *equilibrium considerations only*. Note that the *longitudinal membrane stress*  $N_z$  is *not necessarily constant at this stage* (although its mean value is prescribed). Combining in Sec. 3.4 the generalized plane strain assumption with the hyperelastic constitutive law will allow us to show that  $N_z$  is indeed constant.

### 3.4. The generalized plane strain assumption

The geometry and the loading of the membrane is translation-invariant. It is therefore natural to postulate that its mechanical state is also translation-invariant. In other words, the membrane stress resultants  $N_s$  and  $N_z$  do not depend on  $z$ . It is further assumed that each cross-section remains plane and has the same deformed contour  $\gamma$ . This can only be satisfied by the following transformation

$$s = \Phi_s(S) \quad \text{and} \quad z = \Phi_z(Z). \quad (44)$$

The gradient  $\mathbf{F}$  of the transformation and the right Cauchy–Green strain tensor result from the general expression (6)

$$\mathbf{F} = \frac{d\Phi_s}{dS} \mathbf{t} \otimes \mathbf{T} + \frac{d\Phi_z}{dZ} \mathbf{e}_z \otimes \mathbf{e}_z \quad (45)$$

and

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \left( \frac{d\Phi_s}{dS} \right)^2 \mathbf{T} \otimes \mathbf{T} + \left( \frac{d\Phi_z}{dZ} \right)^2 \mathbf{e}_z \otimes \mathbf{e}_z. \quad (46)$$

Note that  $\mathbf{C}$  is *diagonal*; its principal values are also required to be translation invariant. In particular,  $d\Phi_z/dZ$  must not depend on  $Z$ . This can only be achieved if  $\Phi_z$  is linear:  $\Phi_z(Z) = \lambda_z Z$ , for some constant  $\lambda_z$ . Note that under the plane strain assumption,  $\lambda_z = 1$ . However, owing to the fact that  $N_z \neq 0$  in the present case, it would be incorrect not to allow the membrane to stretch in the longitudinal direction. We therefore make no further assumptions on  $\lambda_z$ , and the transformation of the membrane reads

$$s = \Phi_s(S) \quad \text{and} \quad z = \lambda_z Z, \quad (47)$$

which is usually referred to as the *generalized plane strain assumption* (Zhenye and Shiping, 1990).

Our goal is to express all the membrane equations on the *current* configuration. To this end, we introduce the following stretch  $\lambda_s$

$$\lambda_s(s) = \frac{d\Phi}{dS}[\Psi(s)], \quad \text{where} \quad \Psi = \Phi^{-1}. \quad (48)$$

Note that  $\lambda_s$  thus defined is a function of the arc-length  $s$  on the *deformed* contour  $\gamma$  so that the gradient of the transformation,  $\mathbf{F}$ , and the right Cauchy–Green strain tensor can both be seen as functions of  $s$  only

$$\mathbf{F} = \lambda_s \mathbf{t} \otimes \mathbf{T} + \lambda_z \mathbf{e}_z \otimes \mathbf{e}_z \quad \text{and} \quad \mathbf{C} = \lambda_s^2 \mathbf{T} \otimes \mathbf{T} + \lambda_z^2 \mathbf{e}_z \otimes \mathbf{e}_z. \quad (49)$$

The stretch  $s \mapsto \lambda_s(s)$  measures the elongation of the cross-section. Indeed, we consider a material arc of initial length  $dS$ . Then the current length of this material arc is by definition  $ds = \frac{d\Phi}{dS} dS$ . Inverting, we find

$$dS = \frac{ds}{\lambda_s(s)}, \quad \text{therefore} \quad \mathcal{L}_\Gamma = \oint_{\gamma} \frac{ds}{\lambda_s(s)}. \quad (50)$$

Eq. (50)<sub>2</sub> can be seen as a compatibility condition for the function  $s \mapsto \lambda_s(s)$ ; note that finding  $\lambda_s$  is equivalent to finding  $\Phi$ , since

$$\Psi(s) = \int_0^s \frac{du}{\lambda_s(u)} \quad \text{and} \quad S = \Psi(s). \quad (51)$$

### 3.5. Constitutive equations of translation-invariant membranes

Using the general constitutive equations of Sec. 2.4, we find that  $\mathbf{t}$  and  $\mathbf{e}_z$  are the principal directions of the stress-resultants and

$$N_s = \frac{\mu H}{\lambda_z} \frac{\partial W}{\partial \lambda_s} \quad \text{and} \quad N_z = \frac{\mu H}{\lambda_s} \frac{\partial W}{\partial \lambda_z}. \quad (52)$$

It results from the previous discussions that the stretch  $\lambda_z$  and the membrane stress  $N_s$  are constant along  $\gamma$ . Therefore  $\lambda_s$  and  $N_z$  are *also* constant along  $\gamma$  [see Eq. (52)]. In other words, all mechanical quantities, namely the stretches  $\lambda_s$  and  $\lambda_z$  and the membrane stresses  $N_s$  and  $N_z$  are *constant* along  $\gamma$ . Equilibrium of a translation-invariant membrane therefore reduces to a set of *algebraic* equations. In particular, the compatibility condition (50)<sub>2</sub> and equilibrium equations (43)<sub>2</sub> and (43)<sub>3</sub> deliver

$$\lambda_s = \frac{\mathcal{L}_\gamma}{\mathcal{L}_\Gamma} \quad \text{and} \quad N_z = \frac{p \mathcal{A}_s}{\mathcal{L}_\gamma}. \quad (53)$$

Note that Eq. (53)<sub>2</sub> is an extension of the classical formula  $N_z = p(\pi r^2)/(2\pi r) = p r/2$  for cylindrical pressure vessels.

### 3.6. Summary of the essential formulas

We consider a pressurized membrane which is translation-invariant in the  $z$  direction and constrained to expand inside the space delimited by a few solid planes. All planes are parallel to the  $z$  axis, and contact between these planes and the membrane is assumed frictionless. Solving for the equilibrium of such a system amounts to finding  $p$ ,  $r$ ,  $\lambda_s$ ,  $\lambda_z$ ,  $N_s$  and  $N_z$  and a series of linear segments (contact zones) and circular arcs with radius  $r$ , such that

$$p, r, N_s, N_z \geq 0, \quad \lambda_s, \lambda_z \geq 1 \quad (54)$$



and

$$N_s = p r, \quad N_z = \frac{p \mathcal{A}_\zeta}{\mathcal{L}_\gamma}, \quad (55)$$

$$\frac{N_s}{\mu H} = \frac{1}{\lambda_z} \frac{\partial W}{\partial \lambda_s}, \quad \frac{N_z}{\mu H} = \frac{1}{\lambda_s} \frac{\partial W}{\partial \lambda_z}, \quad (56)$$

$$\lambda_s = \frac{\mathcal{L}_\gamma}{\mathcal{L}_\Gamma}, \quad (57)$$

where Eqs. (55)<sub>1</sub> and (55)<sub>2</sub> express equilibrium, Eqs. (56)<sub>1</sub> and (56)<sub>2</sub> are the constitutive laws and Eq. (57) can be seen as a condition of “geometric compatibility”. Note that for the hyperelastic material of [Anssari-Benam and Bucchi \(2021\)](#), Eqs. (56)<sub>2</sub> and (56)<sub>3</sub> should be specialized as follows

$$\frac{N_s}{\mu H} = n (\lambda_s^4 \lambda_z^2 - 1) \quad \text{and} \quad \frac{N_z}{\mu H} = n (\lambda_s^2 \lambda_z^4 - 1), \quad (58)$$

with

$$n = \frac{(1 - \mathcal{N}) [1 + \lambda_s^2 \lambda_z^2 (\lambda_s^2 + \lambda_z^2 - 9\mathcal{N})]}{\lambda_s^3 \lambda_z^3 (1 - 3\mathcal{N}) [1 + \lambda_s^2 \lambda_z^2 (\lambda_s^2 + \lambda_z^2 - 3\mathcal{N})]}. \quad (59)$$

For *pressure-controlled membranes*,  $p$  is the loading-parameter and the above set of equations [namely: Eqs. (55) to (59)] is sufficient.

For *closed, impermeable membranes*, however, the loading-parameter is the amount of enclosed gas and the above equations must be complemented with its state equation. The membrane is thin enough to ensure that, at equilibrium, the internal temperature is prescribed, equal to the external temperature; in other words, we do not consider the transient temperature changes that might be induced by the mechanical loading. In these conditions, assuming the confined gas to be ideal, Boyle’s law applies : pressure  $\times$  volume = const.

We therefore consider a section of the translation-invariant membrane. In the undeformed configuration, the length in the  $z$  direction is 1, while the area of the cross-section is  $\mathcal{A}_\Sigma$ ; the initial volume of this section is therefore  $1 \times \mathcal{A}_\Sigma$ . In the deformed configuration, the length in the  $z$  direction is  $\lambda_z \times 1$ , while the area of the cross-section is now  $\mathcal{A}_\zeta$ ; the current volume is therefore  $\lambda_z \times 1 \times \mathcal{A}_\zeta$  and Boyle’s law delivers

$$\lambda_z p \mathcal{A}_\zeta = \text{const.} \quad (60)$$

The deformations of a closed, translation-invariant membrane are therefore governed by Eqs. (55) to (60).

### 3.7. Equations of the translation-invariant membrane under the plane strain assumption

In this section, the equations presented in Sec. 3.6 are contrasted with the equations of plane strain equilibrium. In the plane strain assumption, it is assumed that  $\lambda_z = 1$  while  $N_z$  is disregarded. This leads to the following equations, considered by [De Simone and Luongo \(2013\)](#) and [Srivastava and Hui \(2013b\)](#), among others

$$N_s = p r, \quad \mathcal{L}_\gamma = \lambda_s \mathcal{L}_\Gamma, \quad N_s = H \left. \frac{\partial W}{\partial \lambda_s} \right|_{\lambda_z=1} \quad (61)$$

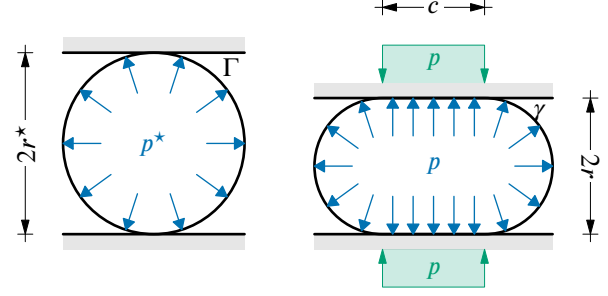


Figure 4: The system under consideration in Sec. 4. At the end of the free inflation phase (left), the membrane is cylindrical with radius  $r^*$  and the inner pressure is  $p^*$ . During the flattening phase (right), the distance between the two plates is decreased from  $2r^*$  to  $2r$ . The length of the contact zone is  $c$ . The contact pressure balances the inner pressure along the two rectilinear segments.

and, for the hyperelastic material of [Anssari-Benam and Bucchi \(2021\)](#),

$$\frac{N_s}{\mu H} = \frac{(1 - \mathcal{N}) (\lambda_s^4 - 1) [1 + \lambda_s^2 (\lambda_s^2 + 1 - 9\mathcal{N})]}{\lambda_s^3 (1 - 3\mathcal{N}) [1 + \lambda_s^2 (\lambda_s^2 + 1 - 3\mathcal{N})]}. \quad (62)$$

For a closed, impermeable membrane, we have the additional equation

$$p \mathcal{A}_\zeta = \text{const.} \quad (63)$$

## 4. Example: a cylindrical membrane squeezed between two rigid, parallel planes

As an illustration of the general theory exposed in Sec. 3, we consider here an initially cylindrical pressurized membrane, compressed between two rigid, parallel planes (see Fig. 4). Our goal is to derive the relationship between the total force applied to the plates and their vertical displacement. This is critical for e.g. the prediction of the gripping force of soft pneumatic bending actuators ([Doreswamy et al., 2022](#); [Sachin et al., 2022a](#)): as an object is gripped, a force is exerted at the tip of the actuator, which tends to squeeze together the various chambers and decreases the curvature of the actuator. Clearly, the same mechanisms are at work as in the simplified system presented here.

This problem was initially analyzed by [De Simone and Luongo \(2013\)](#) under the plane strain assumption, which is questionable owing to the existence of out-of-plane membrane stress resultants. We show that for moderate stretches, the plane strain assumption delivers satisfactory results. For larger stretches, the generalized plane strain assumption ought to be preferred for accurate modelling of pressurized membranes.

The analysis of the membrane is divided in two phases: the free inflation phase is first discussed in Sec. 4.1. Then the flattening of the membrane between the two planes is addressed in Sec. 4.2.

## 4.1. The free inflation phase

During the free inflation phase, the loading parameter is the internal pressure  $p$  and Eq. (60) is not meaningful. The membrane is not in contact with rigid planes and is free to expand as a circular cylinder.

**Generalized plane strain analysis** Eqs. (55) to (59) read

$$N_s = p r, \quad \frac{N_s}{\mu H} = n(\lambda_s^4 \lambda_z^2 - 1), \quad r = \lambda_s R, \quad (64)$$

$$N_z = \frac{p r}{2}, \quad \frac{N_z}{\mu H} = n(\lambda_s^2 \lambda_z^4 - 1). \quad (65)$$

Note that  $N_s$  denotes the hoop membrane stress resultant, while  $\lambda_s$  denotes the hoop stretch. In the above algebraic equations,  $R$  denotes the ‘‘initial radius’’ of the unstretched membrane. Of course, under vanishing internal pressure, the membrane cannot sustain a cylindrical shape :  $R$  should really be understood as an effective initial radius, such that the initial length of the cross-section reads  $\mathcal{L}_\Gamma = 2\pi R$ . The above equations reduce to the following non-linear system with unknowns  $\lambda_s$  and  $\lambda_z$

$$\varpi \lambda_s = n(\lambda_s^4 \lambda_z^2 - 1) \quad \text{and} \quad \frac{1}{2} \varpi \lambda_s = n(\lambda_s^2 \lambda_z^4 - 1), \quad (66)$$

with

$$\varpi = \frac{p R}{\mu H} \quad (\text{dimensionless pressure}). \quad (67)$$

The above algebraic equations are readily solved numerically (using a standard Newton–Raphson solver). Upon elimination of  $\varpi \lambda_s$  between Eqs. (66)<sub>1</sub> and (66)<sub>2</sub>, it is further observed that

$$2\lambda_s^2 \lambda_z^4 - \lambda_s^4 \lambda_z^2 - 1 = 0 \quad \text{and} \quad \lambda_z = \frac{1}{2} \sqrt{\lambda_s^2 + \sqrt{\lambda_s^4 + 8\lambda_s^{-2}}}. \quad (68)$$

For  $\lambda_s \leq 1.4$ , it is therefore found that  $\lambda_z \leq 1.1$  and the plane strain assumption probably remains acceptable (see below). For larger hoop stretches, the longitudinal stretch becomes significantly greater than 1, and can no longer be neglected.

**Plane strain analysis** Eqs. (61) read

$$N_s = p r \quad \text{and} \quad r = \lambda_s R. \quad (69)$$

Upon combination with Eq. (62), we get

$$\varpi = \frac{(1 - \mathcal{N})(\lambda_s^4 - 1)[1 + \lambda_s^2(\lambda_s^2 + 1 - 9\mathcal{N})]}{\lambda_s^4(1 - 3\mathcal{N})[1 + \lambda_s^2(\lambda_s^2 + 1 - 3\mathcal{N})]}. \quad (70)$$

## 4.2. The flattening phase

The state of the membrane at the end of the inflation phase is denoted  $\varpi^*$ ,  $\lambda_s^*$ ,  $\lambda_z^*$ . In the subsequent flattening phase, the amount of gas is constant and Eq. (60) should be considered together with Eqs. (55) to (59). The loading parameter is now the distance between the two parallel planes that constrain the membrane.

The deformed geometry of the membrane,  $\gamma$ , is formed by two half-circles (radius:  $r$ ) and two linear segments (length:  $c$ ) (see Fig. 4)

$$\mathcal{L}_\gamma = 2\pi r + 2c \quad \text{and} \quad \mathcal{A}_\gamma = \pi r^2 + 2c r. \quad (71)$$

It will be convenient to introduce the following dimensionless parameters

$$\rho = \frac{r}{R} \quad \text{and} \quad \chi = \frac{c}{\pi R}. \quad (72)$$

**Generalized plane strain analysis** The (algebraic) equations of the membrane in the flattening phase read

$$N_s = p r, \quad \frac{N_s}{\mu H} = n(\lambda_s^4 \lambda_z^2 - 1), \quad (73)$$

$$N_z = \frac{p r \pi r + 2c}{2 \pi r + c}, \quad \frac{N_z}{\mu H} = n(\lambda_s^2 \lambda_z^4 - 1), \quad (74)$$

$$\lambda_s = \frac{\pi r + c}{\pi R} \quad \lambda_z p r (\pi r + 2c) = \pi \lambda_z^* p^* (r^*)^2, \quad (75)$$

where  $n$  is given by Eq. (59)<sub>3</sub>. In dimensionless form

$$\varpi \rho = n(\lambda_s^4 \lambda_z^2 - 1), \quad \frac{\varpi \rho}{2} \frac{\rho + 2\chi}{\rho + \chi} = n(\lambda_s^2 \lambda_z^4 - 1), \quad (76)$$

$$\lambda_s = \rho + \chi \quad \lambda_z \varpi \rho (\rho + 2\chi) = \varpi^* (\lambda_s^*)^2 \lambda_z^*. \quad (77)$$

**Plane strain analysis** The dimension-free equations of the flattening phase read

$$\varpi \rho = \frac{(1 - \mathcal{N})(\lambda_s^4 - 1)[1 + \lambda_s^2(\lambda_s^2 + 1 - 9\mathcal{N})]}{\lambda_s^3(1 - 3\mathcal{N})[1 + \lambda_s^2(\lambda_s^2 + 1 - 3\mathcal{N})]}, \quad (78)$$

$$\lambda_s = \rho + \chi \quad \text{and} \quad \varpi \rho (\rho + 2\chi) = \varpi^* (\lambda_s^*)^2. \quad (79)$$

## 4.3. Generalized plane strains vs. plane strains

The equations presented above were implemented numerically as described below. *In the free inflation phase*, the simulation is controlled by the hoop stretch  $\lambda_s$ . For each new value  $\lambda_{s,i+1}$ , Newton–Raphson iterations are used to compute the new values of  $\varpi_{i+1}$  and  $\lambda_{z,i+1}$ ; the previous values  $\varpi_i$  and  $\lambda_{z,i}$  are used as initial guesses. *In the flattening phase*, the simulation is controlled by the reduced radius  $\rho$ . For each new value  $\rho_{i+1}$ , Newton–Raphson iterations are used to compute the new values of  $\varpi_{i+1}$ ,  $\lambda_{s,i+1}$ ,  $\lambda_{z,i+1}$  and  $\chi_{i+1}$ ; the previous values  $\varpi_i$ ,  $\lambda_{s,i}$ ,  $\lambda_{z,i}$  and  $\chi_i$  are used as initial guesses.

Pressure-stretch curves are represented in Fig. 5 (continuous lines) for both generalized plane strains (GPS) and plane strains (PS) analyses. For limited stretches  $\lambda_s \leq 1.3$ , the agreement is quite good. It is shown in Appendix F that this is strongly related to the incompressibility of the constitutive material of the membrane. For larger stretches, the predictions of the pressure are widely different. In particular, the plane strain model fails to predict the limit-point and the inflation-jump instabilities (Anssari-Benam et al., 2022).

The membrane is then flattened from two initial configurations:  $\lambda_s^* = 1.3$  and  $\lambda_s^* = 6.0$ . The resulting  $\varpi$  vs.  $\lambda_s$  curves are also plotted on Fig. 5 (dashed curves). Again, excellent agreement is observed when the stretch remains small

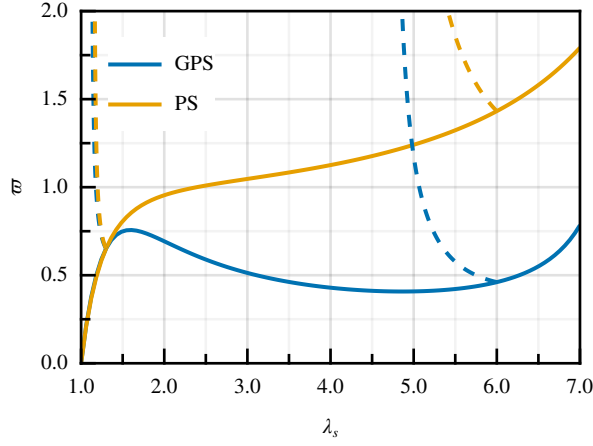


Figure 5: Reduced pressure  $\varpi$  vs. hoop stretch  $\lambda_s$  for the cylindrical membrane discussed in Sec. 4. “GPS”: generalized plane strain; “PS”: plane strain; continuous lines: inflation phase; dashed branches: flattening phase (starting from  $\lambda_s^* = 1.3$  and  $\lambda_s^* = 6.0$ ).

( $\lambda_s^* = 1.3$ ). For  $\lambda_s^* = 6.0$ , the longitudinal stretch  $\lambda_z > 1$  can no longer be neglected.

For engineering applications, prediction of the force–displacement curves is critical. In the present case, the applied force is  $p c$ , while the vertical displacement is  $2r^* - 2r$ . We therefore introduce the following reduced quantities

$$\text{Reduced force} = \frac{p c}{\pi \mu H} = \varpi \chi, \quad (80)$$

$$\text{Reduced displacement} = \frac{2r^* - 2r}{2R} = \lambda_s^* - \rho, \quad (81)$$

which are plotted in Figs. 6 ( $\lambda_s^* = 1.3$ ) and 7 ( $\lambda_s^* = 6.0$ ). For large initial stretches, the plane strain assumption leads to significant errors: even the apparent stiffness of the squeezed membrane (the initial slope of the force–displacement curve) is not correct. As expected, the additional constraint  $\lambda_z = 1$  tends to stiffen the mechanical system.

## 5. Conclusion and perspectives

Pressurized membranes often require to account for geometric and material nonlinearities, including contact. This usually leads to highly non-linear boundary-value problems that can only be solved numerically. For engineering purposes, considering specific situations where the problem simplifies is therefore relevant. De Simone and Luongo (2013) observed that the case of *translation-invariant* membranes under the *plane-strain assumption* is remarkable, as it reduces to a set of two algebraic equations with two scalar unknowns only. However, in many practical applications, the plane-strain assumption is too restrictive as the inner-pressure exerted on the membrane’s caps might induce a large longitudinal stretch. In this paper, we therefore extended the results of De Simone and Luongo (2013) to the case of *translation-invariant* membranes under the *generalized plane-strain assumption*. The membrane might be in frictionless contact with several planes,

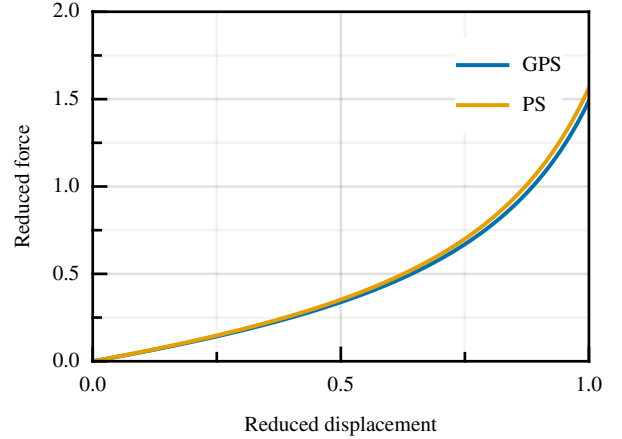


Figure 6: Flattening of a cylindrical membrane, initially stretched to  $\lambda_s^* = 1.3$ . The plot displays the (dimensionless) force–displacement curves. Both generalized plane strain and plane strain analyses are in good agreement.

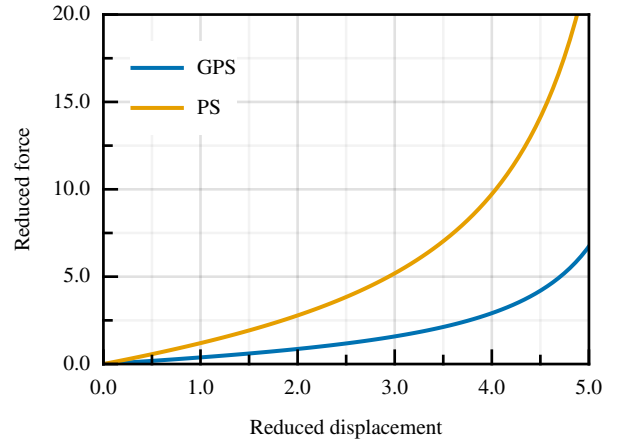


Figure 7: Flattening of a cylindrical membrane, initially stretched to  $\lambda_s^* = 6.0$ . The plot displays the (dimensionless) force–displacement curves. The plane strain analysis leads to significant errors.

all parallel to the longitudinal axis. We show in that case that the boundary-value problem again reduces to a set of algebraic equations, with now *four* unknowns: two stress-resultants and two stretches. Considering an incompressible membrane squeezed between two parallel planes, we showed that neglecting longitudinal effects is acceptable for limited hoop stretches. However, for hoop stretches significantly larger than unity, the generalized plane strain model *must* be preferred to the plane strain assumption.

We have identified a few short term perspectives to this work. First, following [De Simone and Luongo \(2013\)](#), the generalized plane strain assumption should be extended to visco-elastic membranes. The resulting model would reduce to a set of ordinary differential equations involving the longitudinal and hoop stretches and stresses as well as their time derivatives. The required visco-elastic constitutive law is now fully 2D and requires careful formulation ([Firouzi, 2022](#)). Second, contact with friction or adhesion ought to be considered, as was done by [Srivastava and Hui \(2013a\)](#) in the plane strain assumption (see also [Kumar and DasGupta, 2013](#); [Kolesnikov, 2017](#); [Liu et al., 2018](#)); we expect that the resulting system of equations will no longer be algebraic in that case. Third, the general equations that we derived might be used to develop light-weight models of actuators in soft robotics.

A longer term perspective is the analysis of membranes that are rotation-invariant, rather than translation-invariant (e.g. bicycle tires): the membrane is “nearly” translation-invariant when the center of rotation is “far” away. We are hoping to derive an asymptotic model for such situations.

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## A. Proof of Eq. (11)

Using Eq. (2)<sub>2</sub> and observing that: (1)  $\mathbf{a}_3 \cdot \mathbf{N} = \mathbf{0}$  and (2)  $\mathbf{b}$  and  $\mathbf{N}$  are symmetric

$$\begin{aligned}
 \mathbf{a}_3 \cdot \left( \frac{\partial \mathbf{N}}{\partial \xi^\alpha} \cdot \mathbf{a}^\alpha \right) &= \left[ \frac{\partial}{\partial \xi^\alpha} (\mathbf{a}_3 \cdot \mathbf{N}) \right] \cdot \mathbf{a}^\alpha - \left( \frac{\partial \mathbf{a}_3}{\partial \xi^\alpha} \cdot \mathbf{N} \right) \cdot \mathbf{a}^\alpha \\
 &= 0 + (\mathbf{a}_\alpha \cdot \mathbf{b}) \cdot (\mathbf{N} \cdot \mathbf{a}^\alpha) \\
 &= (b_{\alpha\beta} \mathbf{a}^\beta) \cdot (N^{\alpha\gamma} \mathbf{a}_\gamma) \\
 &= b_{\alpha\beta} N^{\alpha\gamma} \delta_\gamma^\beta = b_{\alpha\beta} N^{\alpha\beta} = \mathbf{b} : \mathbf{N}. \quad (82)
 \end{aligned}$$

## B. Constitutive equations of Mooney–Rivlin membranes

For incompressible, Mooney–Rivlin materials, the hyperelastic potential  $\bar{W}$  reads

$$\bar{W}(\lambda_I, \lambda_{II}, \lambda_{III}) = \frac{1}{2} [(I_1 - 3) + \alpha (I_2 - 3)], \quad (83)$$

with

$$\lambda_I \lambda_{II} \lambda_{III} = 1, \quad \text{and} \quad \begin{cases} I_1 = \lambda_I^2 + \lambda_{II}^2 + \lambda_{III}^2 \\ I_2 = \lambda_I^{-2} + \lambda_{II}^{-2} + \lambda_{III}^{-2} \end{cases}. \quad (84)$$

In the above expression,  $\alpha$  (“hardening parameter”) is a material constant that complements the shear modulus  $\mu$ ; the neo-Hookean model is recovered with  $\alpha = 0$ . Plugging  $\lambda_{III} = \lambda_I^{-1} \lambda_{II}^{-1}$  delivers the following reduced potential

$$W(\lambda_I, \lambda_{II}) = \frac{1}{2} [\lambda_I^2 + \lambda_{II}^2 + (\lambda_I \lambda_{II})^{-2} - 3] + \frac{1}{2} \alpha [\lambda_I^{-2} + \lambda_{II}^{-2} + (\lambda_I \lambda_{II})^2 - 3]. \quad (85)$$

Upon substitution into Eqs. (18), the following constitutive equations are found (see e.g. Patil et al., 2014)

$$\frac{N_I}{\mu H} = \left( \frac{\lambda_I}{\lambda_{II}} - \frac{1}{\lambda_I^3 \lambda_{II}^3} \right) (1 + \alpha \lambda_{II}^2), \quad (86)$$

$$\frac{N_{II}}{\mu H} = \left( \frac{\lambda_{II}}{\lambda_I} - \frac{1}{\lambda_I^3 \lambda_{II}^3} \right) (1 + \alpha \lambda_I^2). \quad (87)$$

## C. Proof of Eq. (25)

Eq. (25)<sub>1</sub> is a direct consequence of Eq. (24)<sub>2</sub>, remembering that  $\bar{\mathbf{x}}(\mathcal{L}_\gamma) = \bar{\mathbf{x}}(0)$  since  $\gamma$  is closed

$$\oint_\gamma \mathbf{t} ds = \oint_\gamma \frac{d\bar{\mathbf{x}}}{ds} ds = \bar{\mathbf{x}}(\mathcal{L}_\gamma) - \bar{\mathbf{x}}(0) = \mathbf{0}, \quad (88)$$

while Eq. (25)<sub>2</sub> results from Green’s theorem

$$\begin{aligned} \oint_\gamma \bar{\mathbf{x}} \times \mathbf{t} ds &= \oint_\gamma (\bar{x} \mathbf{e}_x + \bar{y} \mathbf{e}_y) \times \left( \frac{d\bar{x}}{ds} \mathbf{e}_x + \frac{d\bar{y}}{ds} \mathbf{e}_y \right) ds \\ &= \oint_\gamma \left( \bar{x} \frac{d\bar{y}}{ds} - \bar{y} \frac{d\bar{x}}{ds} \right) \mathbf{e}_z ds = \oint_\gamma (x dy - y dx) \mathbf{e}_z \\ &= \int_\zeta \left[ \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right] dx dy \mathbf{e}_z = 2 \int_\zeta dx dy \mathbf{e}_z \\ &= 2\mathcal{A}_\zeta \mathbf{e}_z. \end{aligned} \quad (89)$$

## D. Pressure forces acting on a surface supported on a plane contour

In this appendix, we evaluate the resultant and moment of the pressure forces exerted on a surface (cap) supported on a plane contour. We consider a surface  $\omega$ , supported on the plane contour  $\gamma$  and subjected to a constant inner pressure  $p$ . The outer normal to  $\omega$  is denoted  $\mathbf{n}_\omega$ . The plane surface enclosed by  $\gamma$  is denoted  $\zeta$ ; its normal is  $\mathbf{n}_\zeta$ : it points to  $\omega$  (see Fig. 8).

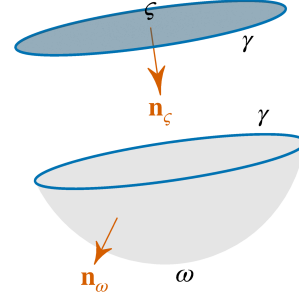


Figure 8: The surface  $\omega$  is supported on the closed, plane contour  $\gamma$ . The outer normal to  $\omega$  is  $\mathbf{n}_\omega$ . The interior of the contour  $\gamma$  is the plane surface  $\zeta$ ; its normal is  $\mathbf{n}_\zeta$ . From the perspective of the closed surface  $\omega \cup \zeta$ ,  $\mathbf{n}_\zeta$  is the *inner* normal. For the sake of clarity, the two surfaces  $\omega$  and  $\zeta$  have been shifted away in the above figure.

The resultant and moment of the pressure forces read

$$\mathbf{R}_{\text{cap}} = \int_{\mathbf{x} \in \omega} p \mathbf{n}_\omega(\mathbf{x}) da = p \int_{\mathbf{x} \in \omega} \mathbf{n}_\omega(\mathbf{x}) da, \quad (90)$$

$$\mathbf{M}_{\text{cap}} = \int_{\mathbf{x} \in \omega} \mathbf{x} \times [p \mathbf{n}_\omega(\mathbf{x})] da = p \int_{\mathbf{x} \in \omega} \mathbf{x} \times \mathbf{n}_\omega(\mathbf{x}) da, \quad (91)$$

where  $da$  is the elemental area, and  $\mathbf{x}$  denotes the radius-vector of the current point on  $\omega$ . Note that the moment  $\mathbf{M}_{\text{cap}}$  is evaluated with respect to the (arbitrary) origin. To evaluate the above integrals, we observe that  $\omega \cup \zeta$  is a closed surface. Therefore, the resultant force and moment exerted on  $\omega \cup \zeta$  by constant pressure forces are both null. In other words (observing that  $\mathbf{n}_\zeta$  is the *inner* normal to  $\omega \cup \zeta$  and does not depend on the observation point)

$$\mathbf{0} = \int_{\mathbf{x} \in \omega} \mathbf{n}_\omega(\mathbf{x}) da + \int_{\mathbf{x} \in \zeta} (-\mathbf{n}_\zeta) da, \quad (92)$$

$$\mathbf{0} = \int_{\mathbf{x} \in \omega} \mathbf{x} \times \mathbf{n}_\omega(\mathbf{x}) da + \int_{\mathbf{x} \in \zeta} \mathbf{x} \times (-\mathbf{n}_\zeta) da, \quad (93)$$

from which it results that

$$\mathbf{R}_{\text{cap}} = p \mathcal{A}_\zeta \mathbf{n}_\zeta \quad \text{and} \quad \mathbf{M}_{\text{cap}} = p \mathcal{A}_\zeta \mathbf{g}_\zeta \times \mathbf{n}_\zeta, \quad (94)$$

where  $\mathcal{A}_\zeta$  denotes the surface area of the plane domain  $\zeta$  delimited by the contour  $\gamma$ , and  $\mathbf{g}_\zeta$  is the radius-vector (indeed, the lever arm) of its center of gravity.

## E. On the external forces applied to the lateral surface of the membrane

In the present appendix, it is shown that the resultant of external (pressure and contact) forces applied to the lateral surface of the membrane are null. We further derive a closed form expression of the longitudinal component of their resultant moment. It is recalled from Sec. 3.2 that the cross-section of the membrane is made of rectilinear segments connected smoothly by circular arcs of radius  $r$  (see Fig. 9).

Let  $n$  be the number of rectilinear segments. Each circular arc subtends with its center an angle  $\beta_1, \dots, \beta_n$ . The central

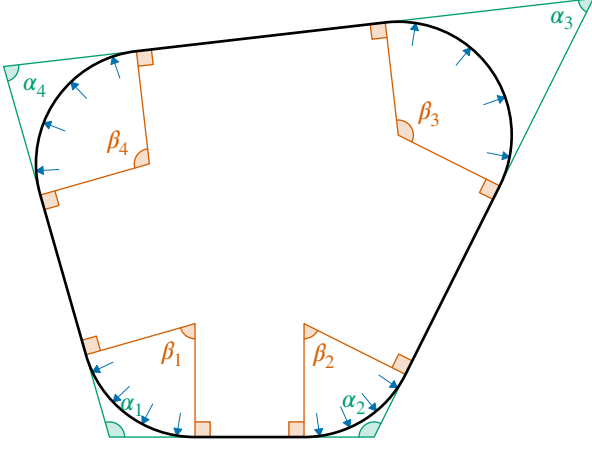


Figure 9: The lateral surface of the membrane (thick line) is a union of  $n$  rectilinear segments connected (smoothly) by circular arcs of radius  $r$  (in the present figure,  $n = 4$ ). Trimming these segments results in a  $n$ -gon with interior angles  $\alpha_1, \dots, \alpha_n$ . The circular arcs that connect the rectilinear segments subtend angles  $\beta_1, \dots, \beta_n$  with  $\beta_k = \pi - \alpha_k$ . Along the rectilinear segments, the inner gas pressure is balanced by the contact forces.

angle  $\beta_k$  is equal to  $\pi - \alpha_k$ , where  $\alpha_k$  is the interior angle of the trimmed polygon. The union of all circular arcs therefore subtends the angle:  $\beta_1 + \dots + \beta_n = n\pi - (\alpha_1 + \dots + \alpha_n)$ . The quantity  $\alpha_1 + \dots + \alpha_n$  is the sum of interior angles of a simple  $n$ -gon: its value is  $(n-2)\pi$  and  $\beta_1 + \dots + \beta_n = n\pi - (n-2)\pi = 2\pi$ , which shows that the translation of these circular arcs form a whole circle.

It has been shown in Sec. 3.2 that contact forces cancel pressure forces in the rectilinear parts of the lateral surface of the membrane. Therefore external (pressure) forces are exerted on the cylindrical parts only. Since the union of all these parts form a closed circular cylinder, the resultant of pressure forces exerted on these cylindrical parts is null. As a conclusion, the resultant of pressure and contact forces on the lateral surface is null:  $\mathbf{R}_{\text{lat}} = \mathbf{0}$ . We now turn to the moment  $\mathbf{M}_{\text{lat}}$  of these forces with respect to the (arbitrary) origin

$$\mathbf{M}_{\text{lat}} = \int_{z_{\min}}^{z_{\max}} \oint_{\tilde{\gamma}} \mathbf{x} \times (f_n \mathbf{n}) ds dz, \quad (95)$$

where  $z_{\min}$  and  $z_{\max}$  are the limits of the section of the lateral surface under consideration, and  $f_n$  is the surface density of external forces applied to the lateral surfaces. In the rectilinear segments,  $f_n = \text{pressure} + \text{contact} = 0$ , while  $f_n = -p$  in the circular segments. Let  $\tilde{\gamma}$  denote the union of all circular arcs of  $\gamma$ . From Eq. (26)<sub>2</sub>

$$\mathbf{M}_{\text{lat}} = -p \int_{z_{\min}}^{z_{\max}} \int_{\tilde{\gamma}} (z \mathbf{e}_z + \bar{\mathbf{x}}) \times \mathbf{n} ds dz \quad (96)$$

and, introducing  $L = z_{\max} - z_{\min}$

$$\begin{aligned} \mathbf{e}_z \cdot \mathbf{M}_{\text{lat}} &= -p \int_{z_{\min}}^{z_{\max}} \int_{\tilde{\gamma}} \mathbf{e}_z \cdot (\bar{\mathbf{x}} \times \mathbf{n}) ds dz \\ &= -pL \int_{\tilde{\gamma}} \mathbf{e}_z \cdot (\bar{\mathbf{x}} \times \mathbf{n}) ds \\ &= pL \int_{\tilde{\gamma}} \bar{\mathbf{x}} \cdot (\mathbf{e}_z \times \mathbf{n}) ds = -pL \int_{\tilde{\gamma}} \bar{\mathbf{x}} \cdot \mathbf{t} ds. \end{aligned} \quad (97)$$

## F. On the role of the Poisson ratio

In this appendix, we show that both pressure-stretch curves coincide in Fig. 5 in the small stretch range by comparing the initial slope of the pressure-stretch curves. These slopes can directly be captured within the framework of the linearized theory of membranes. We consider a general Hooke constitutive material ( $\nu$  is not necessarily equal to  $1/2$ ).

**Under the generalized plane strain assumption** Eqs. (64) and (65) read

$$N_s = pR, \quad \frac{N_s}{EH} = \frac{\varepsilon_s + \nu \varepsilon_z}{1 - \nu^2}, \quad r = (1 + \varepsilon_s)R, \quad (98)$$

$$N_z = \frac{pR}{2}, \quad \frac{N_z}{EH} = \frac{\varepsilon_z + \nu \varepsilon_s}{1 - \nu^2}, \quad (99)$$

where  $\varepsilon_s$  and  $\varepsilon_z$  are the linearized strains and  $E, \nu$  are the Young's modulus and Poisson's ratio of the constitutive material. The above equations readily deliver the linear pressure-strain relationship

$$\frac{pR}{EH} = \frac{\varepsilon_s}{1 - \nu/2}. \quad (100)$$

**Under the plane strain assumption** Eqs. (69) read

$$N_s = pR, \quad \frac{N_s}{EH} = \frac{\varepsilon_s}{1 - \nu^2}, \quad \text{and} \quad r = (1 + \varepsilon_s)R \quad (101)$$

and we get the following linear pressure-strain relationship

$$\frac{pR}{EH} = \frac{\varepsilon_s}{1 - \nu^2}. \quad (102)$$

Note that equations (100) and (102) coincide only when  $\nu = 1/2$  or  $\nu = 0$ , which confirms that for *incompressible membranes*, the differences between the PS and GPS models remain small at small stretch.