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# The Porous Media Schrödinger Equation : Feynman-type Motivation, Well-Posedness and Control Interpretation

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## Abstract

This paper's aim is threefold. First, using Feynman's path approach to the derivation of the classical Schrödinger's equation in [7] and by introducing a slight path (or wave) dependency of the action, we derive a new class of equations of Schrödinger type where the driving operator is no longer the Laplace one but rather of complex porous media-type. Second, using suitable concepts of monotonicity in the complex setting and on appropriate functional spaces, we show the existence and uniqueness of the equation we have previously introduced. Finally, using Fitzpatrick's characterization of maximal monotone operators (cf. [8]), we propose a Brézis-Ekeland type characterization of the solution via a control problem.

**Keywords:** equations of Schrödinger type, Feynman's approach, complex porous media equation, Fitzpatrick's characterization of complex maximal monotone operators, control problem

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Classification: 40.09 Mathematical physics, 40.12 Non-linear PDE, 60.07 Quantum dynamics

## 1 Introduction

The mathematical developments on quantum (mechanics) systems are usually of differential nature in the spirit of Schrödinger's contributions or of algebraic nature in the spirit of Heisenberg's theory. A somewhat different approach based on Hamilton's first principle was proposed in the paper [7] leading to what is sometimes referred to as *Feynman's formula*. The fundamental postulate in the cited reference [7] is the fact that, in the computation of amplitudes, the phase can be taken proportional to the "*time integral of the Lagrangian along a path*" or "*action*" in the nomenclature adopted. The (realization of the) path is then intended in Euclidean sense as a succession of points  $[x_j := x(t_j)]_{-\infty < j < \infty}$  contributing with terms  $S(x_j, x_{j+1})$  in a stationary way. In our developments, we are going to consider a simple form of Lagrangian linear in velocity ( $p$ ) with a potential-related coefficient ( $L(p, x) := \beta(x)p$ ) but, foremost, we ask that the contribution be computed not on a straight line but on the actual wave  $\psi(x, t)$ , thus leading to terms like  $S(\psi(x_j, t_j), \psi(x_{j+1}, t_{j+1})) \approx S(\psi(x_j, t_j), \psi(x_{j+1}, t_j))$ . We wish to point out that, from our point of view, the underlying discrete trajectory is  $[(t_j, x_j := x(t_j))]_{-\infty < j < \infty}$  with an

explicit presence of the time. Furthermore, as already pointed out,  $S$  is computed along the wave traveled.

With these considerations in mind, we derive, in Section 2 a Schrödinger-like PDE, albeit the fact that the governing operator is no longer the Laplace one but the  $\beta$ -induced porous-media nonlinear operator  $\Delta\beta$ , with  $\beta$  being a (possibly) complex valued, complex argument function. To our best knowledge, such equations have not been previously studied and we believe that the aforementioned arguments using Feynman's approach should convince our readers of the interest of such PDEs. Let us also point out that at this level, the considerations are somewhat of an axiomatic nature and  $S$  is not a priori consistent, as it depends on the wave function  $\psi$ . The rigorous study of  $S$  once  $\psi$  is obtained is left for a future work. In the case of Schrödinger's classical operator, we mention the approach in [21, Section 5.5] via  $C_0$ -semigroups. Of course, such arguments for complex porous media operators need a careful study and, as specified, this exceeds the aim we have for this short paper.

In Section 3, we shall adapt some monotonicity tools which are classical in  $\mathbb{R}$ , to our case which needs similar results in  $\mathbb{C}$ . The notion of monotonicity (maximal or  $m$ -monotonic,  $m$ -accretive, etc.) is not new and goes back to [10] in its complex formulation, but it is equally present, with some variations, in the original papers [11] or [4]. Furthermore, monotonic methods have been employed to treat variants of Schrödinger's equation, for instance in [13] or [14] in which Ginzburg-Landau and related equations are dealt with.

In our setting, and much like for usual porous media equations in the real setting, one needs to carefully describe the spaces on which monotonicity for the complex porous media operator can be envisaged and this constitutes the core of Section 3. The choice of spaces, in the spirit of [17], is made with a future treatment of stochastic PDEs of Schrödinger-porous media type in mind. Besides the functional definitions, we link, in Proposition 7 the strict monotonicity of  $\beta$  as a complex function to the maximal monotonicity of the induced porous-media operator. Using these tools, combined with a Galerkin-type approach, we prove the consistency of our equations for strict monotonic  $\beta$ . This constitutes the aim of Section 4. The presence of a further Lipschitz non-linearity is intended for future design of stabilizing controls, either in the area of controllability in the spirit of [9], or for state-constrained design, see [5]. The classical results are not directly applicable and we offer a detailed treatment of the existence.

In the last section (Section 5) we offer a different interpretation of the solution for the Schrödinger-type porous-media equation through an optimal control problem inspired by the variational principle in [3] or again [12]. While in the real case the arguments are connected to Fenchel duality and sub-differential expressions of monotone operators, the complex setting requires to employ variations of Fitzpatrick's characterization of maximal monotonicity, cf. [8]. We offer a detailed treatment of the Brézis-Ekeland-like variational characterization for strictly monotonic  $\beta$  leading to Proposition 19. A different characterization based on the induced porous media operator  $-\Delta\beta$  being also maximal monotonic is hinted at in Remark 20.

To summarize, the main contributions of the paper rely in

- the derivation of the new equation of Schrödinger type driven by porous media complex operators through Feynman-type arguments closely related to the quantum mechanics. We believe this gives a physical and philosophical reason for the study of such equations;
- the variational (or control) interpretation of the solution using Fitzpatrick-type representation of monotone operators. We believe that the independent study of such problems should allow a stand-alone method to prove the existence of solutions;
- for readability and completeness purposes, we provide a (more) direct monotonic approach to the existence of solutions, already available abstract results not applying directly.

## 2 Heuristics on the Model Following Feynman's Approach

### The Lagrangian

To understand the developments hereafter, we begin with recalling some elements constituting Feynman's path approach in [7] as an alternative to Heisenberg-Dirac, see [6] or Schrödinger's approach [18]. The approach can be split into two *postulates*.

1. The most important is the fact that "*the paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of  $\hbar$ )*". This leads, for Feynman, to contributions of a path  $[x(t)]_{t \in \mathbb{R}}$  of type  $\exp\left(\frac{i}{\hbar} S[x(t)]\right)$ . At the same time, the action is to be linked to a Lagrangian depending on the speed and on the position of the point  $L$  and, as such, the action is determined by  $S[x(t)] := \int L(\dot{x}(t), x(t)) dt$ .
2. The remaining postulate requires, as always in quantum analysis, the "superposition of probability amplitudes". Roughly speaking, given measurements  $a$ , resp.  $c$  of events  $A$  and  $C$ , one sums, over intermediate mutually excluding events to get  $\phi_{ac} = \sum_b \phi_{ab} \phi_{bc}$ . When combined with discretized trajectory  $[x_j := x(t_j)]_{j \in \mathbb{Z}}$ , this leads (see [7, Eq. (9)]) to  $\phi(R) \approx \int_R \exp\left[\frac{i}{\hbar} \sum_{-\infty < j < \infty} S(x_{j+1}, x_j)\right] \dots \frac{dx_{j+1}}{A} \frac{dx_j}{A} \dots$ , given a region  $R$ . The parameter  $A$  is a normalization one.

As a consequence of these postulates, the *wave function* computed on the  $t_k$ -non anticipating sub-region  $R'$  is given by

$$\psi(x_k, t) \approx \int_{R'} \exp\left[\frac{i}{\hbar} \sum_{-\infty < j < k} S(x_{j+1}, x_j)\right] \dots \frac{dx_{k-2}}{A} \frac{dx_{k-1}}{A} \frac{1}{A}.$$

Then Schrödinger's original equation is obtained with the usual Lagrangian in a movement-against- $V$ -potential i.e.  $L(\dot{x}, x) := \frac{m(\dot{x})^2}{2} - V(x)$ .

Let us give **another way of interpreting these arguments**. Let us first fix a Lipschitz-continuous (for now) real function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  and denote by  $\dot{\beta}$  its  $L^\infty(\mathbb{R})$  almost everywhere derivative. In our case, we shall consider the  $L$  Lagrangian of the following form

$$L^0(p, x) := \dot{\beta}(x)p.$$

Following the idea from Feynman [7], one can describe the path of a free particle by a straight-line and therefore the energy on  $[t_j, t_{j+1} = t_j + \varepsilon]$  is approximated (provided  $\dot{\beta}$  is continuous) as

$$\begin{aligned} (1) \quad S^\varepsilon(x_{j+1}, x_j) &= \frac{\varepsilon}{2} \left( L\left(\frac{x_{j+1} - x_j}{\varepsilon}, x_{j+1}\right) + L\left(\frac{x_{j+1} - x_j}{\varepsilon}, x_j\right) \right) \\ &= \frac{\dot{\beta}(x_{j+1}) + \dot{\beta}(x_j)}{2} (x_{j+1} - x_j), \end{aligned}$$

or, in a simpler formulation, as

$$(2) \quad S^\varepsilon(x_{j+1}, x_j) = \varepsilon L\left(\frac{x_{j+1} - x_j}{\varepsilon}, \frac{x_{j+1} + x_j}{2}\right) = \dot{\beta}\left(\frac{x_{j+1} + x_j}{2}\right) (x_{j+1} - x_j).$$

Feynman mentions such forms of the Lagrangian with emphasis on the difference in scales, see [7, Page 376]. Particular emphasis is put on (2) for the symmetry in the expected Hamiltonian.

In Feynman's interpretation, given a path  $(t_k, x_k)_{k \in \mathbb{Z}}$  the transition

$$(\dots, t_{j-1}, x_{j-1}, t_j, x_j) \longrightarrow (\dots, t_{j-1}, x_{j-1}, t_j, x_j, t_{j+1}, x_{j+1})$$

is Markovian, homogeneous (independent of the number of transitions) stationary (only depending on the time available  $t_k - t_{k-1}$  but not on  $t_{k-1}$ ) and governed by a non-random  $S$  as before,

hence leading to

$$\begin{aligned}
\psi(x_k, t_k) &: = \int e^{\frac{i}{\hbar} \sum_{j=-\infty}^{k-1} S_{j+1}(t_{j+1}, x_{j+1}, t_j, x_j, \dots)} \frac{dx_{k-1}}{A} \frac{dx_{k-2}}{A} \dots \\
&= \mathbb{E} \left[ e^{\frac{i}{\hbar} \sum_{j=-\infty}^{k-1} S_{j+1}(t_{j+1}, X_{j+1}, t_j, X_j, \dots)} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ e^{\frac{i}{\hbar} \sum_{j=-\infty}^{k-1} S_{j+1}(t_{j+1}, X_{j+1}, t_j, X_j, \dots)} \mid \mathcal{F}_{k-1} \right] \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ e^{\frac{i}{\hbar} S_{j+1}(t_k, x_k, t_{k-1}, X_{k-1}, \dots)} \mid \mathcal{F}_{k-1} \right] \psi(X_{k-1}, t_{k-1}) \right] \\
&= \mathbb{E} \left[ e^{\frac{i}{\hbar} S^{t_k - t_{k-1}}(x_k, X_{k-1})} \psi(X_{k-1}, t_{k-1}) \right],
\end{aligned}$$

where  $A$  is a factor whose value we shall determine later and  $\hbar$  is the reduced Planck constant.

Although obvious, it is maybe worth mentioning that  $X_j$  have  $\frac{dx_j}{A}$  densities if  $X_j \leq k-1$  while  $X_k = x_k$  is fixed and  $\mathcal{F}$  is the naturally induced filtration associated to these random variables. Furthermore,  $S$  can depend on the number of transition, hence the sub-index  $j+1$ . We also emphasize that the last equality only holds for the stationary, Markovian case which is considered in the original Feynman computations.

### Complex developments

Since the previous developments are done in  $\mathbb{R}$ , we are trying to slightly generalize in two directions: by changing the space to  $\mathbb{C}$  and by considering another form of path dependence.

The reader is invited to recall that the complex space  $\mathbb{C}^n$  has a Hilbert structure with the usual scalar product  $\langle z_1, z_2 \rangle = \sum_{1 \leq j \leq n} \overline{z_{1,j}} z_{2,j}$  which is (left-)sesquilinear. As usual,  $z = (z_j)_{1 \leq j \leq n}$  is a column  $n$ -dimensional vector over  $\mathbb{C}$ .

We consider  $\beta := \beta_1 + i\beta_2$ ,  $\beta_k : \mathbb{R}^2 \rightarrow \mathbb{R}$  which is (complex-)differentiable.

We now turn to a model that assumes perhaps the simplest form of *path dependence* in the formulation (2), i.e.

$$(3) \quad S(z, s, z') = \beta' \left( \frac{\psi(z', s) + \psi(z, s)}{2} \right) (\psi(z, s) - \psi(z', s)).$$

Of course, in this case, the derivative is computed in the complex sense and we indicated this by writing  $\beta'$ . Heuristically speaking, the reader is invited to notice the following.

1. The distance is not taken in Euclidean metric between positions  $x$  and  $y$  but along the wave  $\psi(\cdot, s)$  with the time  $s$  fixed;
2. Stationarity is no longer enforced as the "starting" time  $s$  enters the expression;
3. Since  $\psi$  represents a memory of the path (as it is integrated along past events), the computations are, in some sense, path-dependent;
4. one is actually looking into a fixed point of

$$(4) \quad \begin{cases} S(x, t, y) := \beta' \left( \frac{\psi(y, t) + \psi(x, t)}{2} \right) (\psi(x, t) - \psi(y, t)); \\ \psi(x_0 = x, t_0 = t) := \lim_{\varepsilon \rightarrow 0} \int e^{\frac{i}{\hbar} \sum_{n \geq 0} S(x_n, t_n, x_{n+1})} \frac{dx_{n+1}}{A} \frac{dx_{n+2}}{A} \dots, \\ 0 < \varepsilon := \sup_{n \in \mathbb{Z}} (t_{n+1} - t_n); \end{cases}$$

5. the integrals can be taken along the same points as in the aforementioned formula for  $\psi(x_k, t_k)$  but involving copies  $\tilde{X}$  independent of  $X$ .

We emphasize that, at this point, we are not going to prove the consistency of (4) but merely use this in order to deduce the associated *Schrödinger-type equation*.

We have

$$(5) \quad \psi(x, t + \varepsilon) = \int e^{-\frac{i}{\hbar} \beta' \left( \frac{\psi(x + \xi, t) + \psi(x, t)}{2} \right) [\psi(x + \xi, t) - \psi(x, t)]} \psi(x + \xi, t) \frac{d\xi}{A}.$$

It is expected that the normalization  $A$  is of order of  $\varepsilon^{1/2}$  and so is (the support of)  $\xi$ . For simplicity, this support is taken  $[-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$  with the obvious (uniform) normalization  $A := 2\sqrt{\varepsilon}$ . Since we are going to rely on approximations up to order  $\varepsilon$ , we ignore the terms over  $\xi^2$ . Baring this in mind, we have

$$\begin{aligned}
(6) \quad & \psi(x, t + \varepsilon) \\
& \simeq \int_{[-\sqrt{\varepsilon}, \sqrt{\varepsilon}]} \exp \left( \frac{i}{\hbar} \left\{ -\frac{\beta''(\psi(x, t))}{2} [\partial_x \psi(x, t) \xi]^2 - \beta'(\psi(x, t)) \left[ \partial_x \psi(x, t) \xi + \frac{\xi^2}{2} \partial_{x^2}^2 \psi(x, t) \right] \right\} \right) \\
& \quad \times \left[ \psi(x, t) + \partial_x \psi(x, t) \xi + \frac{\xi^2}{2} \partial_{x^2}^2 \psi(x, t) \right] \frac{d\xi}{2\sqrt{\varepsilon}} \\
& \simeq \int_{[-\sqrt{\varepsilon}, \sqrt{\varepsilon}]} \left( 1 - \frac{i}{2\hbar} \Delta \beta(\psi(x, t)) \xi^2 - \frac{i}{\hbar} \nabla \beta(\psi(x, t)) \xi - \frac{1}{2\hbar^2} (\nabla \beta(\psi(x, t)))^2 \xi^2 \right) \\
& \quad \times \left[ \psi(x, t) + \partial_x \psi(x, t) \xi + \frac{\xi^2}{2} \partial_{x^2}^2 \psi(x, t) \right] \frac{d\xi}{2\sqrt{\varepsilon}}.
\end{aligned}$$

We get

$$(7) \quad \varepsilon \partial_t \psi(x, t) \simeq \frac{\varepsilon}{6} \left[ \psi(x, t) \left( -\frac{i}{\hbar} \Delta \beta(\psi(x, t)) - \frac{1}{\hbar^2} (\nabla \beta(\psi(x, t)))^2 \right) + \Delta \psi(x, t) - 2 \frac{i}{\hbar} \nabla \beta(\psi(x, t)) \nabla \psi(x, t) \right].$$

We now set  $\beta(\varepsilon) = \beta^0(\cdot)$  and set  $\phi := \log \psi$  to get

$$\begin{aligned}
(8) \quad & \partial_t \phi(x, t) \simeq \\
& \frac{1}{6} \left[ -\frac{i}{\hbar} \Delta \beta^0(\phi(x, t)) - \frac{1}{\hbar^2} (\nabla \beta^0(\phi(x, t)))^2 + \Delta \phi(x, t) + (\nabla \phi(x, t))^2 - 2 \frac{i}{\hbar} \nabla \beta^0(\phi(x, t)) \nabla \phi(x, t) \right] \\
& = \frac{1}{6} \left[ \Delta \left( -\frac{i}{\hbar} \beta^0 + Id \right) (\phi(x, t)) + \left( \nabla \left( -\frac{i}{\hbar} \beta^0 + Id \right) (\phi(x, t)) \right)^2 \right] \\
& = \left[ \Delta \tilde{\beta}(\phi(x, t)) + \gamma (\nabla \tilde{\beta}(\phi(x, t)))^2 \right],
\end{aligned}$$

where  $\tilde{\beta} = \frac{1}{6} \left( -\frac{i}{\hbar} \beta^0 + Id \right)$  and  $\gamma = 6 > 0$ .

The fundamental example one has in mind is  $\tilde{\beta}(z) = \kappa i z$  and a classical change allows one to obtain Schrödinger's equation from (8). Indeed, in this case, we are dealing with

$$\partial_t \phi(x, t) = \kappa i \Delta \phi(x, t) - \gamma \kappa^2 (\nabla \phi(x, t))^2,$$

and by setting  $\Phi := \frac{e^{i\kappa\gamma\phi}}{\kappa\gamma}$ , one computes

$$\partial_t \Phi = i\kappa\gamma \Phi \partial_t \phi; \quad \nabla \Phi = i\kappa\gamma \Phi \nabla \phi; \quad \Delta \Phi = i\kappa\gamma \Phi \Delta \phi - \kappa^2 \gamma^2 \Phi (\nabla \phi)^2,$$

which amounts to

$$(9) \quad \partial_t \Phi(x, t) = i\kappa \Delta \Phi = \Delta \tilde{\beta}(\Phi(x, t)),$$

which is the classical Schrödinger's equation.

This trick and observation is also used for [16], see also [15, Section 4] to treat an equation of form (8) (with the usual Laplace operator instead of porous media-one) from

$$(10) \quad \begin{cases} \partial_t \phi(x, t) = \Delta \beta(\phi(x, t)) + f(t), & \text{on } \mathcal{O} \times (0, T), \\ \phi(0, \cdot) = \phi_0, & \text{on } \mathcal{O} \times \{0\}, \\ \phi = 0, & \text{on } \Gamma := \partial \mathcal{O}. \end{cases}$$

### 3 Monotonicity Approach in $\mathbb{C}$ to the Existence and Uniqueness of Solution

In this section we shall present an adaptation to  $\mathbb{C}$  of different notions and results which are necessary to study equation (10) using a monotonicity approach.

**Definition 1** 1. A function  $\beta : \mathbb{C} \rightarrow \mathbb{C}$  is said to be monotone if there exists  $\alpha \in \mathbb{R}_+$  such that

$$\Re \langle \beta(z) - \beta(z'), z - z' \rangle_{\mathbb{C}} \geq \alpha |z - z'|^2, \quad \forall z, z' \in \mathbb{C}.$$

When  $\alpha > 0$ , the function is called strictly monotone.

2. Given a complex linear vector space  $(V, \|\cdot\|_V)$  whose dual is denoted by  $V^*$ , the functional  $B : D(B) \subset V \rightarrow V^*$  is said to be monotone if there exists  $\alpha \in \mathbb{R}_+$  such that

$$\Re \langle B(h) - B(h'), h - h' \rangle_{V^*, V} \geq \alpha \|h - h'\|_V^2, \quad \forall h, h' \in D(B),$$

with  $D$  classically denoting the domain.

**Remark 2** Note that if  $\beta$  is monotone, then

$$\begin{aligned} |z - z' + a(\beta(z) - \beta(z'))|^2 &\geq 2a \Re \langle \beta(z) - \beta(z'), z - z' \rangle \\ &\geq |z - z'|^2 + 2a\alpha \|z - z'\|^2 \\ &\geq |z - z'|^2, \end{aligned}$$

for all  $a \in \mathbb{R}_+$ . Due to the uniform convexity of  $\mathbb{C}$  the two notions coincide in this framework (see [10, (M) and (M')]).

One can easily see in the following examples a way of understanding the notions above.

**Example 3** 1. For every  $p \geq 1$ , the function  $\beta(z) := i|z|^{p-1}z$  is monotone in the sense of the previous definition. Indeed,

$$\begin{aligned} &\Re \left( \overline{\beta(z_1) - \beta(z_2)} (z_1 - z_2) \right) \\ &= \Re \left[ \left( -i \left( |z_1|^{p-1} \Re z_1 - |z_2|^{p-1} \Re z_2 \right) - \left( |z_1|^{p-1} \Im z_1 - |z_2|^{p-1} \Im z_2 \right) \right) (z_1 - z_2) \right] \\ &= \left( |z_1|^{p-1} \Re z_1 - |z_2|^{p-1} \Re z_2 \right) (\Im z_1 - \Im z_2) - \left( |z_1|^{p-1} \Im z_1 - |z_2|^{p-1} \Im z_2 \right) (\Re z_1 - \Re z_2) \\ &= 0. \end{aligned}$$

2. With the same argument, one can actually show that  $\beta(z) := \tilde{\beta}(|z|)z$  is monotone, for  $\tilde{\beta} : \mathbb{R}_+ \rightarrow \mathbb{C}$  such that  $\Re \tilde{\beta}$  has non-negative values. If  $\Re \tilde{\beta}$  is bounded from below away from 0, then the monotonicity is strict (i.e.  $\alpha = \inf_{x \in \mathbb{R}_+} \Re \tilde{\beta}(x) > 0$ ).

In the present work we are essentially interested in a complex form of the porous media operator. For this reason, in connection to the afore-mentioned functions  $\beta$ , we will introduce the porous media operator  $\Delta \beta$ .

Given an open bounded set  $\mathcal{O} \subset \mathbb{R}^n$ , we introduce the following Sobolev spaces which are necessary for the construction of the complex porous media operator:

- $\mathbb{L}^2 = \{ \phi : \mathcal{O} \rightarrow \mathbb{C} : \Re \phi, \Im \phi \in \mathbb{L}^2(\mathcal{O}; \mathbb{R}) \}$  with the scalar product

$$\langle \phi, \psi \rangle_{\mathbb{L}^2} := \int_{\mathcal{O}} \overline{\phi(\xi)} \psi(\xi) d\xi, \quad \phi, \psi \in \mathbb{L}^2,$$

and the induced Euclidean norm.

- $H_0^1 := \{ \phi : \mathcal{O} \rightarrow \mathbb{C} : \Re \phi, \Im \phi \in H_0^{1,2}(\mathcal{O}; \mathbb{R}) \}$ , for the classical Sobolev spaces of order 1 in  $\mathbb{L}^2(\mathcal{O}; \mathbb{R})$  where Dirichlet boundary conditions are enforced. The scalar product is the usual one involving  $\mathbb{C}$ -linear spaces i.e.

$$\langle \phi, \psi \rangle_{H_0^1} := \int_{\mathcal{O}} \langle \nabla \phi(\xi), \nabla \psi(\xi) \rangle_{\mathbb{C}^n} d\xi, \quad \phi, \psi \in H_0^1.$$

Furthermore, the induced norm will be denoted by  $\|\cdot\|_{H_0^1}$ .

The reader is invited to note that this is possible because  $\mathcal{O}$  is assumed to be bounded and due to Poincaré's inequality (otherwise,  $\langle \phi, \psi \rangle_{\mathbb{L}^2}$  should be added). One considers  $(H_0^1)^*$  to be the dual of  $H_0^1$ . The duality (such that  $(H_0^1, \mathbb{L}^2, (H_0^1)^*)$  is a *Ghelfand triple* when  $\mathbb{L}^2$ 's dual is identified with itself via Riesz's principle) yields

$$\langle \phi, \psi \rangle_{(H_0^1)^*, H_0^1} = \int_{\mathcal{O}} \overline{\phi(\xi)} \psi(\xi) d\xi.$$

As a consequence, and by using integration-by-parts arguments, whenever  $\phi, \psi \in H_0^1$ , one has

$$\langle -\Delta \phi, \psi \rangle_{(H_0^1)^*, H_0^1} = \int_{\mathcal{O}} \langle \nabla \phi(\xi), \nabla \psi(\xi) \rangle_{\mathbb{C}^n} d\xi = \langle \phi, \psi \rangle_{H_0^1}.$$

As a consequence,  $-\Delta : H_0^1 \rightarrow (H_0^1)^*$  provides an isometric isomorphism. The Hilbert structure on  $(H_0^1)^*$  is given by

$$\langle \Delta \phi, \Delta \psi \rangle_{(H_0^1)^*} = \langle \phi, \psi \rangle_{H_0^1}, \quad \forall \phi, \psi \in H_0^1.$$

Now one can identify  $(H_0^1)^*$  and  $H_0^1$  via  $(-\Delta)^{-1}$  and use this identification in order to obtain the Gelfand triple

$$(11) \quad V := \mathbb{L}^2 \subset (H_0^1)^* \subset V^*.$$

Throughout the paper, we will keep this notation in order not to perturb our readers when reading  $(\mathbb{L}^2)^*$  as defined above. It is then immediate that  $-\Delta$  extends to an isometry from  $V$  to  $V^*$  such that

$$(12) \quad \langle -\Delta \phi, \psi \rangle_{V^*, V} = \langle \phi, \psi \rangle_{\mathbb{L}^2},$$

for all  $\phi, \psi \in V = \mathbb{L}^2$ . All these arguments follow in the same way as their real-spaces analogous. The interested reader is invited to take a look at [17, Example 4.1.7, Example 4.1.11, Lemmas 4.1.12; 4.1.13] for the real-valued settings.

**Definition 4** For an operator  $\beta : \mathbb{C} \rightarrow \mathbb{C}$  which is monotone and has linear growth i.e.

$$(13) \quad \exists c \in \mathbb{R}_+ \text{ s.t. } |\beta(z)| \leq c(1 + |z|), \quad \forall z \in \mathbb{C}.$$

we can properly define the complex porous media operator by

$$A := \Delta \beta : V \rightarrow V^*, \quad A(\phi) := \Delta \beta(\phi), \quad \forall \phi \in \mathbb{L}^2.$$

One can check that  $-A := -\Delta \beta$  is monotone on  $(H_0^1)^*$ . Indeed,

$$\begin{aligned} \Re \left( \langle -A(\phi) + A(\psi), \phi - \psi \rangle_{V^*, V} \right) &= \Re \left( \int_{\mathcal{O}} \langle \beta(\phi(\xi)) - \beta(\psi(\xi)), \phi(\xi) - \psi(\xi) \rangle_{\mathbb{C}} d\xi \right) \\ &= \int_{\mathcal{O}} \Re \langle \beta(\phi(\xi)) - \beta(\psi(\xi)), \phi(\xi) - \psi(\xi) \rangle_{\mathbb{C}} d\xi \\ &\geq \alpha \int_{\mathcal{O}} |\phi(\xi) - \psi(\xi)|^2 d\xi = \alpha \|\phi - \psi\|_{\mathbb{L}^2}^2. \end{aligned}$$

Note that strict monotonicity of  $\beta$  implies strict monotonicity of  $-A$ .

**Example 5** If one assumes the particular form  $\beta(z) = \tilde{\beta}(|z|)z$ , then  $\beta(\bar{z}) = \tilde{\beta}(|z|)\bar{z}$  such that

$$(14) \quad \overline{A(\phi)} = \overline{\Delta \beta(\phi)} = \Delta \Re \beta(\phi) - i \Delta \Im \beta(\phi) = \Delta \tilde{\beta}(|\phi|) (\Re \phi - i \Im \phi) = A(\bar{\phi}),$$

for all  $\phi \in \mathbb{L}^2$ .



**Remark 6** 1. A complex function  $\beta$  can be identified with a function  $\begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  and the monotonicity requirement is just

$$(15) \quad \left\langle \begin{pmatrix} \Re\beta(x_1, x_2) \\ \Im\beta(x_1, x_2) \end{pmatrix} - \begin{pmatrix} \Re\beta(y_1, y_2) \\ \Im\beta(y_1, y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle_{\mathbb{R}^2} \geq 0$$

thus reducing to the monotonicity on  $\mathbb{R}^2$ . It is for this reason that the functional arguments (on monotonicity, for instance) can be conducted on real Hilbert spaces.

For instance, if  $\beta$  is continuous and strictly monotone, then it has full range  $\mathbb{C}$ . Indeed, using the above identification,  $\begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix}$  is maximal monotone on the real Hilbert space  $\mathbb{R}^2$  (one can apply, for instance, [1, Theorem 2.4]). On the other hand, strict monotonicity implies coercivity in  $\mathbb{R}^2$ . As such (e.g. [1, Corollary 2.2] for a version of the Minty-Browder Theorem),  $\begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix}$  has full range  $\mathbb{R}^2$ , or, equivalently,  $\beta$  has full range  $\mathbb{C}$ .

2. The isometry between  $V$  and  $V^*$  exhibited in the previous example is surjective, see [17, Remark 4.1.14]. In this particular case, the duality map between  $V$  and  $V^*$ , denoted by  $J$  in [1] (and  $F$  in the introduction of [10]) is actually given by the Riesz isomorphism involving  $(-\Delta)$  and one is actually using the monotonicity condition [10, (M')] in Definition 1.

We claim that the operator  $-A := -\Delta\beta$  is actually maximal monotone (or, if one prefers, *m-accretif*) when  $\beta$  is strictly monotone, that is, we have the following result.

**Proposition 7** If  $\beta$  is strictly monotone and has at most linear growth (13) as a  $\mathbb{C}$ -valued function, then the domain of  $(J - A)^{-1}$  is the entire space  $V^*$  (or, equivalently,  $-A$  is maximal monotone on  $(V, V^*)$ ).

**Proof.** Although classical, this amounts to show that  $(J - \Delta\beta)(u) = v$  admits a solution  $u \in \mathbb{L}^2(\mathcal{O})$  as soon as  $v \in V^*$  (see [1, Theorem 2.2]).

On one hand, it is easy to see that  $\beta^{-1}$  is monotone and Lipschitz-continuous on  $\mathbb{C}$  (please refer to Remark 6 for the domain). Indeed, the strict monotonicity yields

$$\Re \langle x - y, \beta^{-1}(x) - \beta^{-1}(y) \rangle \geq \alpha |\beta^{-1}(x) - \beta^{-1}(y)|^2,$$

with  $\alpha > 0$  yielding  $|\beta^{-1}(x) - \beta^{-1}(y)| \leq \frac{1}{\alpha} |x - y|$ . As a consequence,  $\beta^{-1}$  provides a (Lipschitz-)continuous, bounded and monotone operator on  $V$  and  $J\beta^{-1}$  gives a continuous, bounded and monotone operator from  $V$  to  $V^*$ .

The duality mapping is given by  $-\Delta$  is maximal monotone (see, for instance, [1, Page 28] for the real case, from which the complex one follows immediately). It follows that the operator  $J\beta^{-1} - \Delta$  is maximal monotone (as in [1, Corollary 2.1]). Furthermore,  $J\beta^{-1} - \Delta$  is coercive since

$$\frac{\langle u, (J\beta^{-1} - \Delta)u \rangle_{V, V^*}}{\|u\|_V} \geq \left(1 + \frac{1}{\alpha}\right) \|u\|_V.$$

As such (see [1, Corollary 2.2]), the equation  $(J\beta^{-1} - \Delta)u = v$  admits a solution  $u \in V$  whenever  $v \in V^*$  and  $\bar{u} := \beta(u)$  satisfies  $(J - \Delta\beta)(u) = v$ . ■

**Remark 8** 1. The reader is invited to note that  $z \mapsto \beta(z) := i|z|^{p-1}z$  is also surjective.

Indeed, if  $w \in \mathbb{C}$ , and  $w \neq 0$ , then  $z := \frac{-iw}{|w|^{\frac{p-1}{p}}}$  provides the solution to  $\beta(z) = w$  (the case  $w = 0$  leads to  $z = 0$ ). As such, one can define  $\beta^{-1}(z) := \frac{-iz}{|z|^{\frac{p-1}{p}}}$  and extend it to be 0 at  $z = 0$  by continuity. The continuity of the  $\mathbb{L}^q(\mathcal{O})$ -induced operator requires a more careful consideration of the power  $q$  and at least the Gelfand triple should be changed.

2. It is clear that if  $\beta$  is monotone,  $z \mapsto \alpha\Re(z) + \beta(z)$  is strictly monotone as soon as  $\alpha > 0$ .

## 4 Existence and Uniqueness of the Solution

In this section we shall proof the existence and uniqueness of the solution to an equation constructed as equation (10) in the sense of the definition below.

**Definition 9** Let  $X_0 \in \mathbb{L}^2(\mathcal{O})$  be an initial condition. We call  $X$  a strong solution to equation

$$(16) \quad \begin{cases} dX(t) = (A(X(t)) + f(X(t))) dt, & t \in (0, T), \\ X(0) = X_0, \end{cases}$$

if the following conditions are satisfied

- $X \in C\left([0, T]; (H_0^1)^*(\mathcal{O})\right) \cap \mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O}))$ ;
- $\beta(X(s)) \in H_0^1(\mathcal{O})$ , a.s. on  $[0, T]$ ,  $\int_0^\cdot \Delta\beta(X(s)) ds \in \mathbb{L}^\infty\left([0, T]; (H_0^1)^*(\mathcal{O})\right)$ ;
- $X(t) = X_0 + \int_0^t \Delta\beta(X(s)) ds + \int_0^t f(X(s)) ds, \quad \forall t \in [0, T]$ .

We shall prove existence and uniqueness of the solution for the equation we are interested in under the following assumptions.

- Assumption 10**
1. Let  $\beta : \mathbb{C} \rightarrow \mathbb{C}$  be a Lipschitz continuous, strictly monotone function such that  $\beta(0) = 0$
  2. We consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  a function which is Lipschitz continuous in  $\mathbb{L}^2(\mathcal{O})$  and also in  $(H_0^1)^*$ .

**Remark 11** For explicit examples of such functions  $f$ , the reader is referred to [5, Example 11]. The adaptation to the complex case is straight-forward.

The reader is invited to note that, the classical method based on the Yosida approximation from [2] does not apply in our case (please take a look at [2, eq. (2.52)])

We can give now the main result of this section

**Theorem 12** Under the Assumptions 10, for each initial condition  $X_0 \in \mathbb{L}^2(\mathcal{O})$ , we have a unique solution to equation (16) in the sense of Definition 9.

**Proof.**

We know by Proposition 7 that under the assumptions on  $\beta$ , the operator  $-A := -\Delta\beta$  is maximal monotone on  $(V, V^*)$ .

Indeed, the maximal monotonic case goes back to [10] and relies on Yosida approximations  $J_n := (I - \frac{1}{n}A)^{-1}$  (see [10, Sections 2 and 3]). We will prefer here a Galerkin-type approach. To this purpose, let  $(\lambda_j, e_j)$  be the eigen-values/eigen-functions associated to  $-\Delta$  on  $(H_0^1)(\mathcal{O}; \mathbb{R})$  (with Dirichlet boundary conditions). These functions are taken in  $\mathbb{L}^2(\mathcal{O}; \mathbb{R})$  and form an orthonormal basis in  $(H_0^1)(\mathcal{O}; \mathbb{R})$ . It is straightforward that  $\tilde{e}_j := \frac{1}{\sqrt{\lambda_j}} e_j$  form an orthonormal basis in  $\mathbb{L}^2(\mathcal{O}; \mathbb{R})$ . For  $n \geq 1$ , we set

$$\Pi_n \phi = \Pi_n \Re \phi + i \Pi_n \Im \phi.$$

The reader will note that if  $x \in \mathbb{L}^2(\mathcal{O}; \mathbb{R})$  is real-valued, then

$$\begin{aligned} \Pi_n x &= \sum_{1 \leq j \leq n} \langle x, e_j \rangle_{(H_0^1)^*(\mathcal{O})} e_j = \sum_{1 \leq j \leq n} \frac{1}{\lambda_j} \langle x, -\Delta e_j \rangle_{(H_0^1)^*(\mathcal{O})} e_j = \sum_{1 \leq j \leq n} \frac{1}{\lambda_j} \langle x, e_j \rangle_{\mathbb{L}^2(\mathcal{O})} e_j \\ &= \sum_{1 \leq j \leq n} \langle x, \tilde{e}_j \rangle_{\mathbb{L}^2(\mathcal{O})} \tilde{e}_j. \end{aligned}$$

In other words,  $\Pi_n$  gives the same projection when one looks at  $\mathbb{L}^2(\mathcal{O})$  and  $(H_0^1)^*(\mathcal{O})$  (by definition, also for  $x \in \mathbb{L}^2(\mathcal{O})$  complex).

It is by now standard that  $-\Pi_n A$  provides a Lipschitz operator both in  $(H_0^1)^*(\mathcal{O})$  and in  $\mathbb{L}^2(\mathcal{O})$ .

The equation

$$(17) \quad dX_n(t) = (\Pi_n A(X_n(t))) + \Pi_n f(X_n(t)) dt, \quad t \in (0, T), \quad X_n(0) = \Pi_n X_0,$$

is well-posed (through standard Lipschitz arguments) in  $C\left([0, T]; (H_0^1)^*(\mathcal{O})\right) \cap \mathbb{L}^2\left([0, T]; \mathbb{L}^2(\mathcal{O})\right)$ . It is also straightforward that this solution lives in  $\Pi_n\left((H_0^1)^*(\mathcal{O})\right)$  i.e.

$$\Pi_n X_n = X_n.$$

Furthermore, with the use of a differential formula (for instance, by taking a look at [10, Lemma 1.3]), one gets

$$\begin{aligned} & \|X_n(t)\|_{(H_0^1)^*(\mathcal{O})}^2 - 2\Re \int_0^t \langle \Pi_n A(X_n(s)), X_n(s) \rangle_{(H_0^1)^*(\mathcal{O})} ds \\ &= \|\Pi_n X_0\|_{(H_0^1)^*(\mathcal{O})}^2 + 2 \int_0^t \Re \langle \Pi_n f(X_n(s)), X_n(s) \rangle_{(H_0^1)^*(\mathcal{O})} ds. \end{aligned}$$

We recall that  $-A = -\Delta\beta$ , leading to<sup>1</sup>

$$\begin{aligned} \langle -\Pi_n A(X_n), X_n \rangle_{(H_0^1)^*(\mathcal{O})} &= \langle \Pi_n \beta(X_n), X_n \rangle_{\mathbb{L}^2(\mathcal{O})} \\ &= \langle \beta(X_n), X_n \rangle_{\mathbb{L}^2(\mathcal{O})} \geq \|X_n\|_{\mathbb{L}^2(\mathcal{O})}^2, \end{aligned}$$

and, by the hypotheses above,  $\left| \Re \langle f(x), x \rangle_{(H_0^1)^*(\mathcal{O})} \right| \leq C \left(1 + \|x\|_{(H_0^1)^*(\mathcal{O})}^2\right)$ .

As a consequence,

$$\begin{aligned} (18) \quad & \|X_n(t)\|_{(H_0^1)^*(\mathcal{O})}^2 + 2\Re \int_0^t \langle \beta(X_n(s)), X_n(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds \\ & \leq \|X_0\|_{(H_0^1)^*(\mathcal{O})}^2 + C \int_0^t \left(1 + \|X_n(s)\|_{(H_0^1)^*(\mathcal{O})}^2\right) ds, \end{aligned}$$

and, with a simple application of Gronwall's inequality, we get

$$(19) \quad \begin{cases} (i) & \|X_n(t)\|_{(H_0^1)^*(\mathcal{O})}^2 \leq e^{Ct} \left(Ct + \|X_0\|_{(H_0^1)^*(\mathcal{O})}^2\right), \\ (ii) & \Re \int_0^t \langle \beta(X_n(s)), X_n(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds \leq \frac{e^{Ct}}{2} \left(Ct + \|X_0\|_{(H_0^1)^*(\mathcal{O})}^2\right), \\ (iii) & \int_0^t \|X_n(s)\|_{\mathbb{L}^2(\mathcal{O})}^2 ds \leq \frac{e^{Ct}}{2} \left(Ct + \|X_0\|_{(H_0^1)^*(\mathcal{O})}^2\right), \end{cases}$$

for all  $t \geq 0$ .

As a consequence, and along some subsequence, still denoted  $n$  for simplicity,

$$(20) \quad \begin{cases} X_n \text{ converges to } X \text{ weakly } * \text{ in } \mathbb{L}^\infty\left([0, T]; (H_0^1)^*(\mathcal{O})\right); \\ X_n \text{ converges to } X \text{ weakly in } \mathbb{L}^2\left([0, T]; \mathbb{L}^2(\mathcal{O})\right); \\ \beta(X_n) \text{ converges to some } Y \text{ weakly in } \mathbb{L}^2\left([0, T]; \mathbb{L}^2(\mathcal{O})\right). \end{cases}$$

Similarly, whenever  $n, m \geq 1$ , by noting that  $\Pi_n = \Pi_{n \vee m} \Pi_n$ , one has (assuming, without loss of generality,  $m \geq n$ ),

$$\begin{aligned} (21) \quad & \|X_n(t) - X_m(t)\|_{(H_0^1)^*(\mathcal{O})}^2 \\ & + 2\Re \int_0^t \langle \Pi_m \beta(X_n(s)) - \Pi_m \beta(X_m(s)), X_n(s) - X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds \\ & = 2\Re \int_0^t \langle \Pi_n f(X_n(s)) - X_m(s), X_n(s) - X_m(s) \rangle_{(H_0^1)^*(\mathcal{O})} ds \\ & + 2\Re \int_0^t \langle (\Pi_m - \Pi_n) \beta(X_n(s)), X_n(s) - X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds. \end{aligned}$$

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<sup>1</sup>Without loss of generality, in order to avoid constant complications, we assume that  $\Re \langle \beta(z), z \rangle \geq |z|^2$  (i.e. the strict monotonicity constant of  $\beta$  is 1).

Then, reasoning as before, one gets

$$\begin{aligned} & \|X_n(t) - X_m(t)\|_{(H_0^1)^*(\mathcal{O})}^2 + 2\Re \int_0^t \langle \beta(X_n(s)) - \beta(X_m(s)), X_n(s) - X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds \\ & \leq C \int_0^t \|X_n(s) - X_m(s)\|_{(H_0^1)^*(\mathcal{O})}^2 ds + 2\Re \int_0^t \langle (\Pi_m - \Pi_n)\beta(X_n(s)), X_n(s) - X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})} ds. \end{aligned}$$

The reader is reminded that  $\Pi_n X_n = X_n$  (and similarly for  $X_m$ ) to see that

$$\langle (\Pi_m - \Pi_n)\beta(X_n(s)), X_n(s) - X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})} = -\langle (1 - \Pi_n)\beta(X_n(s)), X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})}.$$

Using Gronwall's inequality,

$$(22) \quad \begin{aligned} & \|X_n(t) - X_m(t)\|_{(H_0^1)^*(\mathcal{O})}^2 + 2 \int_0^t \|X_n(s) - X_m(s)\|_{\mathbb{L}^2(\mathcal{O})}^2 ds \\ & \leq \int_0^T \mathbf{1}_{s \leq t} e^{Ct} \alpha_{n,m}(s) ds, \end{aligned}$$

where

$$(23) \quad \begin{cases} \alpha_{n,m}(s) & := -2e^{-Cs} \Re \langle (1 - \Pi_n)\beta(X_n(s)), X_m(s) \rangle_{\mathbb{L}^2(\mathcal{O})}; \\ \alpha_n(s) & := -2e^{-Cs} \Re \langle (1 - \Pi_n)\beta(X_n(s)), X(s) \rangle_{\mathbb{L}^2(\mathcal{O})}. \end{cases}$$

First, let us take  $t = T$  in (22) and note that, for  $n$  fixed, by the weak convergence of  $X_m$  in  $\mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O}))$ ,

$$\lim_{m \rightarrow \infty} \int_0^T e^{CT} \alpha_{n,m}(s) ds = \int_0^T e^{CT} \alpha_n(s) ds.$$

Fatou's Lemma applied to (22) with  $t = T$  yields

$$2 \int_0^T \|X_n(s) - X(s)\|_{\mathbb{L}^2(\mathcal{O})}^2 ds \leq \int_0^T \mathbf{1}_{s \leq t} e^{Ct} \alpha_n(s) ds.$$

Owing to (19) (iii) and to the fact that  $(1 - \Pi_n)X$  converges strongly to 0 in  $\mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O}))$ , one deduces that  $\int_0^T \mathbf{1}_{s \leq t} e^{Ct} \alpha_n(s) ds$  converges to 0 as  $n \rightarrow \infty$ , which shows that

$$X_n \text{ converges strongly in } \mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O})) \text{ to } X.$$

As a consequence of the Lipschitz-property of  $\beta$ ,  $Y = \beta(X)$ . For our readers' sake, we emphasize that this equally implies that  $-\Delta Y$  can be identified with  $-\Delta\beta(X)$  as elements in  $\mathbb{L}^2([0, T]; (\mathbb{L}^2(\mathcal{O}))^*)$ .

Furthermore, it follows that  $\alpha_{n,m}$  converges in  $\mathbb{L}^1([0, T]; \mathbb{R})$  to 0 implying that  $X$  is also the  $C([0, T]; (H_0^1)^*(\mathcal{O}))$ -limit of  $X_n$  (as  $n \rightarrow \infty$ ). By writing down the limiting integral equality with respect to  $(\mathbb{L}^2(\mathcal{O}))^*$ , one has the last condition in Definition 9. It then follows that  $\int_0^t \Delta\beta(X(s)) ds \in (H_0^1)^*(\mathcal{O})$ , Lebesgue-almost surely on  $[0, T]$ , which in turn implies the second condition in Definition 9.

The uniqueness of the solution is obtained directly from Gronwall's inequality, by using the monotonicity and the Lipschitz property of the function  $f$ . ■

**Remark 13** *Note that the previous solution is strong in  $(H_0^1)^*(\mathcal{O})$  and therefore it can be considered also in the form*

$$X(t) = X_0 + \Delta \int_0^t \beta(X(s)) ds + \int_0^t f(X(s)) ds, \quad \forall t \in [0, T],$$

*which corresponds to a variational formulation in  $(H_0^1)^*(\mathcal{O})$ . From the monotonicity of the operator  $\beta$  we can directly obtain uniqueness also for the variational formulation.*

*This form will appear in the following optimal control formulation, since the necessary Fitzpatrick function characterizes, in our complex context, functions  $\beta$  which are merely monotone, while the strong formulation, even for real cases, usually holds only in the strictly monotone framework.*

## 5 Control Interpretation

### 5.1 Some Elements of Representation for Maximal Monotonic Operators

We consider a real Banach space  $\mathbb{V}$  and we recall that whenever  $\Phi : \mathbb{V} \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex and lower semicontinuous (l.s.c.), then

$$\partial\Phi(x) := \{p \in \mathbb{V}^* : \Phi(y) \geq \Phi(x) + \langle p, y - x \rangle_{\mathbb{V}^*, \mathbb{V}}, \forall y \in \mathbb{V}\},$$

denotes the *subdifferential* of  $\Phi$  at  $x \in \mathbb{V}$ .

In the real case, the porous media equation can be seen as a control problem using the fact that  $\beta$ , when strictly monotone (hence  $-\Delta\beta$  maximal monotone on  $(\mathbb{L}^2(\mathcal{O}; \mathbb{R}), [\mathbb{L}^2(\mathcal{O}; \mathbb{R})]^*)$ ) can be identified with the subdifferential of some  $\Phi$  and  $\Phi(x) + \Phi^*(p) = \langle p, x \rangle_{\mathbb{R}}$  only when  $p \in \partial\Phi(x)$ .

As for the definition of monotonicity, the one of subdifferentials can be extended to complex Banach spaces by following the spirit of the Remark 6.

**Definition 14** *If  $\mathbb{V}$  is a complex Banach space and  $\Phi : \mathbb{V} \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex and lower semicontinuous,*

$$\partial\Phi(x) := \{p \in \mathbb{V}^* : \Phi(y) \geq \Phi(x) + \Re \langle p, y - x \rangle_{\mathbb{V}^*, \mathbb{V}}, \forall y \in \mathbb{V}\},$$

denotes the subdifferential of  $\Phi$  at  $x \in \mathbb{V}$ .

**Example 15** *Now, the simplest example one has in mind is the Schrödinger operator corresponding, up to a non-negative constant to  $z \mapsto iz$ . We will actually reason on  $\beta(z) = \lambda z + iz$ ,  $\forall z \in \mathbb{C}$ , with  $\lambda \in \mathbb{R}_+$  capturing strict monotonicity as well if  $\lambda > 0$ . In this framework, if we want to represent  $\beta = \partial\Phi$  as given by the previous definitions, this leads to*

$$\Phi(y) - \Phi(x) \geq \Re \langle \lambda x + ix, y - x \rangle.$$

Taking (the lower limit as)  $x \rightarrow 0$  leads to  $\Phi(y) \geq \liminf_{z \rightarrow 0} \Phi(z)$ . Let us consider  $y := r(1 - \lambda i)x$ , with  $r \in \mathbb{R}$ . The previous inequality reads

$$\Phi(x) \leq \Phi(r(1 - \lambda i)x) - r|x|^2 \Re[(\lambda - i)(1 - \lambda i)] = \Phi(r(1 - \lambda i)x),$$

and taking (the lower limit as)  $r \rightarrow 0$  leads to  $\liminf_{z \rightarrow 0} \Phi(z) \geq \Phi(x)$  which implies  $\Phi$  is constant and this leads to a contradiction. It follows that even for the simplest cases one has to find a cleverer way to "represent"  $\beta$ .

In the complex case, given a monotone  $\beta$ , we define, inspired by [8] (see also [19]),

$$(24) \quad \mathcal{F}_\beta(z_1, z_2) := \Re \langle z_1, z_2 \rangle - \inf_{u \in \mathbb{C}} \Re \langle z_1 - u, z_2 - \beta(u) \rangle$$

We shall use in the control interpretation of the problem the following properties of the Fitzpatrick function.

**Proposition 16** *Let  $\beta$  be a monotone function. The following properties hold true.*

1.  $\mathcal{F}_\beta : \mathbb{C}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper lower semi-continuous convex function.
2. For all  $z_1, z_2 \in \mathbb{C}$ ,  $\mathcal{F}_\beta(z_1, z_2) \geq \Re \langle z_1, z_2 \rangle$ .
3. For all  $z_1 \in \mathbb{C}$ ,  $\beta(z_1)$  is the unique  $z_2 \in \mathbb{C}$  such that  $\mathcal{F}_\beta(z_1, z_2) = \Re \langle z_1, z_2 \rangle$ .

**Proof of Proposition 16.** The reader is invited to note that

$$\mathcal{F}_\beta(z_1, z_2) = \sup_{u \in \mathbb{C}} \Re \left[ \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \beta(u) \\ u \end{pmatrix} \right\rangle_{\mathbb{C}^2} - \langle u, \beta(u) \rangle_{\mathbb{C}} \right],$$

(i.e. a supremum over a family of linear functions), thus providing a real-valued lower semi-continuous convex function on  $\mathbb{C}^2$ .

Since  $\beta$  is continuous and monotone,  $\begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  is a maximal monotone operator, where, by abuse of notation, we have identified  $\Re\beta(z) = \Re\beta\left(\begin{pmatrix} \Re z \\ \Im z \end{pmatrix}\right)$  when  $z \in \mathbb{C}$  (and similarly for  $\Im\beta$ ). By definition,  $\mathcal{F}_\beta$  can be identified with the function  $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R} \cup \{+\infty\}$  given by

$$g(\Re z_1, \Im z_1, \Re z_2, \Im z_2) := \left\langle \begin{pmatrix} \Re z_1 \\ \Im z_1 \end{pmatrix}, \begin{pmatrix} \Re z_2 \\ \Im z_2 \end{pmatrix} \right\rangle - \inf_{u \in \mathbb{R}^2} \left\langle \begin{pmatrix} \Re z_1 \\ \Im z_1 \end{pmatrix} - u, \begin{pmatrix} \Re z_2 \\ \Im z_2 \end{pmatrix} - \begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix}(u) \right\rangle.$$

This corresponds to the Fitzpatrick function associated to  $\begin{pmatrix} \Re\beta \\ \Im\beta \end{pmatrix}$  (see [19, Eq. (1.2)]) and the assertions are merely re-interpretations of [19, Eq. (1.3)] for the case of Hilbert spaces  $\mathbb{R}^2$  (where the dual is identified with  $\mathbb{R}^2$ ). ■

In the spirit of the Brezis-Ekeland variational principle, we can construct the following two optimal control problem which are equivalent to the existence result.

## 5.2 A Variational Formulation

In connection with the equation (16), we formally define the following  $u$ -controlled dynamics

$$(25) \quad dX(t) = (\Delta u(t) + f(X(t))) dt.$$

With respect to the aforementioned dynamics, we consider the control functional

$$(26) \quad J(x, u) := \int_0^T \int_{\mathcal{O}} (\mathcal{F}_\beta(X^{x,u}(t), u(t)) - \Re \langle X^{x,u}(t), u(t) \rangle) d\xi dt.$$

In the notion of solution, one seeks an integral expression of type

$$X^{x,u}(t) = x + \Delta \int_0^t u(s) ds + \int_0^t f(X^{x,u}(s)) ds,$$

with  $u$  taking its values in  $H_0^1$  and such that  $\int_0^\cdot u(s) ds \in \mathbb{L}^\infty([0, T]; H_0^1)$  (please take a look at Definition 9). The reader is invited to note the fact that, by the point 3 of the Proposition 16, an optimal pair  $(X^*, u^*)$  of the previous problem which satisfies also  $J(X^*, u^*) = 0$  is a solution to (16).

In order to ensure the well-posedness of the problem above, we shall first write the following equivalent formulation

$$(27) \quad J(X^{x,v}, v) := \int_0^T \int_{\mathcal{O}} (\mathcal{F}_\beta(X^{x,v}(t), \partial_t v(t)) - \Re \langle X^{x,v}(t), \partial_t v(t) \rangle) d\xi dt,$$

which is subject to

$$(28) \quad X^{x,v} = x + \Delta v_t + \int_0^t f(X^{x,v}(s)) ds,$$

with a simple notation  $v(t) := \int_0^t u(s) ds$ .

The requirement on the control is now simplified, and amounts to  $v \in \mathbb{L}^\infty([0, T]; H_0^1)$  which, impacts the solution  $X^{x,u}$  with the requirement that  $X^{x,u} \in \mathbb{L}^\infty([0, T]; (H_0^1)^*)$ . The Lipschitz requirement on  $f$  with respect to  $(H_0^1)^*$  takes care of the remaining integral term in (28). Furthermore, by a slight abuse of notation and in preparation of the precise statement of our problem, we no longer have a functional of the initial condition  $x$ , but rather of an element in  $\mathbb{L}^\infty([0, T]; (H_0^1)^*)$ .

However, we still have to deal with the scalar product taken in  $\mathbb{L}^2(\mathcal{O})$  and this amounts to imposing that  $X^{x,v} \in \mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O}))$  respectively  $v \in W^{1,2}([0, T]; \mathbb{L}^2(\mathcal{O}))$ .

This equally gives us the consistency of the term involving Fitzpatrick's functional under the assumptions 10. Indeed, since  $\beta$  is assumed to be strictly monotone, 0 at 0, and Lipschitz-continuous, it follows that

$$|\beta(u)| \leq C|u|, \text{ and } \Re \langle u, \beta(u) \rangle \geq c|u|^2,$$

for some positive real constants  $c, C$ . This leads to

$$\begin{aligned} \Re (\langle u, z_2 \rangle + \langle \beta(u), z_1 \rangle - \langle u, \beta(u) \rangle) &\leq \frac{c}{2} |u|^2 + \frac{1}{2c} |z_2|^2 + \frac{c}{2} |u|^2 + \frac{C^2}{2c} |z_1|^2 - c|u|^2 \\ &= \frac{1}{2c} |z_2|^2 + \frac{C^2}{2c} |z_1|^2. \end{aligned}$$

As a consequence,  $\mathcal{F}_\beta(X^{x,u}(t), \partial_t v(t)) \in \mathbb{L}^1([0, T] \times \mathcal{O}; \mathbb{R})$ .

Finally, invoking the Lipschitz regularity of  $f$ , this time with respect to  $\mathbb{L}^2(\mathcal{O})$ , gives us the consistency in (27) in  $\mathbb{L}^2(\mathcal{O})$ .

As a consequence, one can concentrate on the following.

### Problem 17

$$\begin{aligned} \text{Minimize } J(y, v) &:= \int_0^T \int_{\mathcal{O}} \mathcal{F}_\beta(y(t), \partial_t v(t)) + \Re \langle v(t), f(y(t)) \rangle d\xi dt \\ &+ \frac{1}{2} \|v(T)\|_{H_0^1(\mathcal{O})}^2 - \Re \int_{\mathcal{O}} \left\langle v(T), y_0 + \int_0^T f(y(s)) ds \right\rangle d\xi, \\ \text{over } y &\in \mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^\infty([0, T]; (H_0^1)^*), \\ v &\in W^{1,2}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^\infty([0, T]; H_0^1), \\ \text{subject to } \Delta v(t) &= y(t) - y_0 - \int_0^t f(y(s)) ds. \end{aligned}$$

**Remark 18** *To understand this, the reader is invited to note that if  $y$  satisfies the state constraint, then, with standard integration by parts, one gets*

$$\begin{aligned} &\int_0^T \int_{\mathcal{O}} \Re \langle \partial_t v(s), y(t) \rangle d\xi dt \\ &= \int_0^T \int_{\mathcal{O}} \Re \left\langle \partial_t v(t), y_0 + \int_0^t f(y(r)) dr \right\rangle d\xi dt + \int_{\mathcal{O}} \Re \langle \partial_t v(t), \Delta v(t) \rangle d\xi dt \\ &= \Re \int_{\mathcal{O}} \left[ \left\langle v(T), y_0 + \int_0^T f(y(s)) ds \right\rangle - \int_0^T \langle v(t), f(y(t)) \rangle dt \right] d\xi - \frac{1}{2} \|\nabla v(T)\|_{\mathbb{L}^2(\mathcal{O})}^2, \end{aligned}$$

such that the cost functional is exactly

$$J(y, v) := \int_0^T \int_{\mathcal{O}} \mathcal{F}_\beta(y(t), \partial_t v(t)) - \Re \langle \partial_t v(t), y(t) \rangle d\xi dt.$$

As such,  $J \geq 0$  (according to the second assertion in Proposition 16) and the 0 value is attained for  $\partial_t v(t) = \beta(y(t))$ , that is for  $y$  being the solution  $X^{y_0, v}$  (which follows from the third assertion in Proposition 16). These consideration put together and by invoking Theorem (12) give the following Brézis-Ekeland characterization.

**Proposition 19** *We ask Assumption 10 to hold true. Then, for every  $y_0 \in \mathbb{L}^2(\mathcal{O})$ , the problem 17 has a unique optimal solution  $(y^*, v^*)$  such that*

- (i)  $y^* \in \mathbb{L}^2([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^\infty([0, T]; (H_0^1)^*)$ ;
- (ii)  $v^* \in W^{1,2}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^\infty([0, T]; H_0^1)$ , and

(iii)  $\Delta v^*(t) = y^*(t) - y_0 - \int_0^t f(y^*(s))ds$  almost everywhere.

Furthermore,

(iv)  $J(y^*, v^*) = 0$  and

(v)  $\partial_t v^* = \beta(y^*)$ , and  $y^* = X^{y_0, \beta(y^*)}$  is the unique solution to (16) starting from  $X_0 = y_0$ .

We end this subsection with some remarks that motivate the consideration of the control problem, besides the generalization of known variational principles.

**Remark 20** 1. *The above formulation is also valid for the classic Schrödinger operator and, more general, for merely monotone  $\beta$ . Indeed, the solvability of the control problem 17 together with a null optimal value as in point (iv) implies the existence of a solution to (16) in the distributional sense as pointed out in Remark 13.*

2. *The Problem 17 is classically formulated (semi-continuity, convexity, well-posedness of terms), the only aspect missing being the coercitivity. However, it is known that Fitzpatrick's function  $\mathcal{F}_\beta$  is not the unique one representing the maximal monotone function  $\beta$  (see, for instance [20]). This approach can equally provide a solution to special cases when  $\beta$  is monotonic but not strictly monotonic; we recall that its maximal monotonicity is still guaranteed in  $\mathbb{C}$ .*

3. *Again the reference [20, Section 5] formally deals with stability of equations and this can be extended to our porous-media complex setting. However, the classical regularizations of  $\beta$  to guarantee strict monotonicity i.e.  $\beta_\varepsilon := \beta + \varepsilon \Re Id$  followed by the natural Fitzpatrick choice of representatives  $\mathcal{F}_{\beta_\varepsilon}$  is not covered by the abstract assumptions related to  $\Gamma$ -convergence given as examples in [20, Section 5]. This is not entirely surprising, partly due to the previous remark.*

4. *Again under the assumption of strict monotonicity on  $\beta$ , another choice of maximal operator (see Proposition 7) is  $-A = -\Delta\beta$  on  $(V, V^*) := (\mathbb{L}^2(\mathcal{O}), (\mathbb{L}^2(\mathcal{O}))^*)$  (see the aforementioned result and discussions preceding it). In this case too, Fitzpatrick's function  $\mathcal{F}_{-A}$  can be defined by setting*

$$\mathcal{F}_{-A}(x, x^*) := \sup_{(a, a^*) \in \text{gr}(-A)} \Re \left[ \langle a, x^* \rangle_{(V, V^*)} + \langle x, a^* \rangle_{(V, V^*)} - \langle a, a^* \rangle_{(V, V^*)} \right],$$

where  $\text{gr}$  denotes the graph (see [19, Eq. (1.2)]). The properties are similar to the ones exhibited in Proposition 16 and another variational problem equivalent to the consistency of (16) can be formulated.

## Submission statement

The work presented here has not been published previously, it is not under consideration for publication elsewhere.

The publication is approved by all authors and by the responsible authorities where the work was carried out. If accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder.

## Declaration of interest

The authors have no competing interest to declare.

## Declaration of generative AI in scientific writing

The paper makes no use of generative AI.

## Author contributions

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## Data Availability Statement

No new data were created or analyzed in this study. Data sharing is not applicable to this article.

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